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Subject	Strength of Materials (SOM)	Section	
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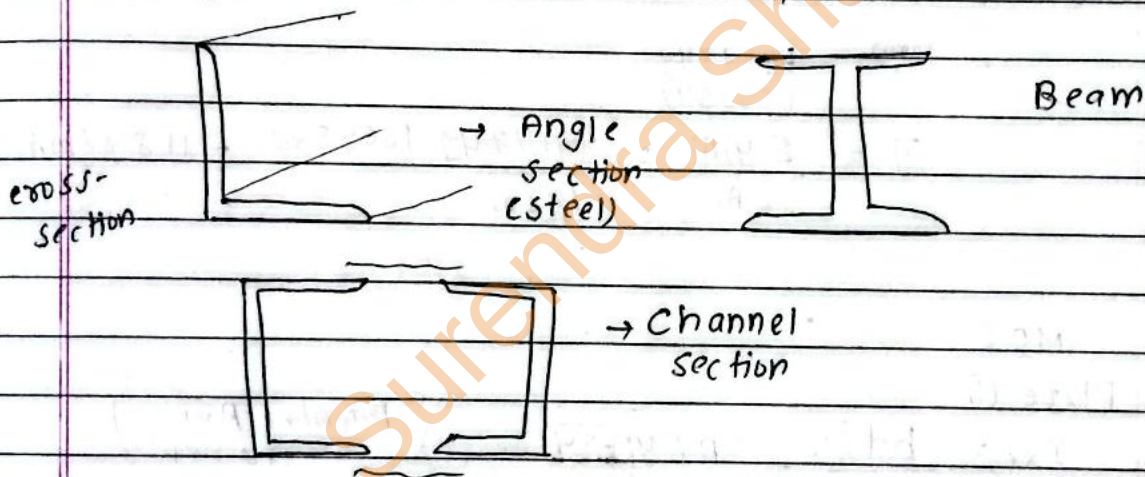
Chapters

- (1) Beams & Frames
- (2) MOI, Product of Inertia
- (3) Stress & Strain → Deformable bodies
- (4) Analysis of Stress
- (5) Thin walled vessels → Eg: LPG Gas
- (6) Torsion → Rotation about own longitudinal axis
- (7) Flexure Theory → Bending forces
- (8) Column Theory

Reference:-

- (1) G.B.M
- (2) B.C. Punmia
- (3) R.K. Rajput

Chapter-2 : MOI, Product of Inertia



→ For same cross-section area of rectangle & I-Beam, I-Beam has greater depth

Beam Design: → Greater depth ⇒ Higher MOI

↓
Resist Bending
on application of
vertical load.

→ depth = $(1.5-2) \times \text{width}$

→ Since, ~~the~~ I-section has larger depth than rectangular beam. ∴ Steel Beam are generally I-section.

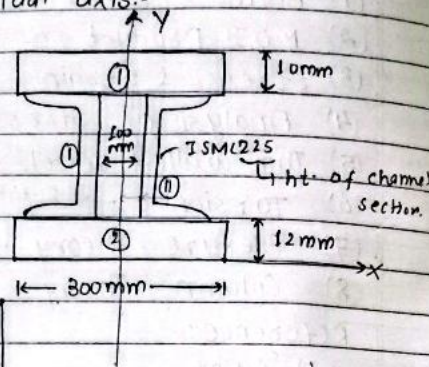
(Q) Find MOI about centroidal axis:

→ Soln:-

(Ref. IS 808)

ISM C → Indian Standard
Metric Channel

Step (I): Location of centroid



Component	y_i	A_i	$y_i A_i$
Plate (1)	242	3000	726000
Plate (2)	6	3600	21600
channel (1)	124.5	3301	410974.5
channel (2)	124.5	3301	410974.5
	$\Sigma y_i A_i$	ΣA_i	
	13801	13801	1569549
	$\bar{y} = \frac{\Sigma y_i A_i}{\Sigma A_i}$		$= 118.88 \text{ mm}$

Step: (II)

MOI

(a) Plate (1)

$$I_{x_c} = \frac{bd^3}{12} + A(y_i - \bar{y})^2 \quad \left\{ \begin{array}{l} \text{Parallel Axis} \\ \text{Theorem} \end{array} \right\}$$

$$= \frac{300 \times 10^3}{12} + 3000(242 - 118.88)^2$$

$$= 45500603.2 \text{ mm}^4$$

$$I_{y_c} = \frac{db^3}{12} + A(\bar{x}_i - \bar{x})^2$$

$$= \frac{300^3 \times 10}{12} + 3000(0 - 0)^2$$

$$= 22500000 \text{ mm}^4$$

(b) channel (1)

$$I_{x_c} = 2694.6 \times 10^4 + 3301(124.5 - 118.88)^2$$

$$= 27050260.1 \text{ mm}^4$$

$$I_{y_c} = 187.2 \times 10^4 + 3301(50 + 20.3)^2$$

$$= 18185839.09 \text{ mm}^4$$

(c) plate (2)

$$I_{x_c} = \frac{10^3 \times 300}{12} + 3600(6 - 0)^2$$

$$= 25000 \text{ mm}^4 = 45895819.84 \text{ mm}^4$$

$$= 45914019.84 \text{ mm}^4$$

$$I_{y_c} = \frac{10 \times 300^3}{12} + 3600(0 - 0)^2$$

$$= 27000000 \text{ mm}^4$$

(d) Channel (2)

$$I_{x_c} = 27050260.1 \text{ mm}^4$$

$$I_{y_c} = 18185839.09 \text{ mm}^4$$

Thus,

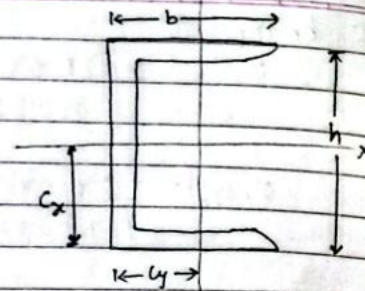
Total MOI about centroidal axis:

$$I_{x_c} = 145515143.2 \text{ mm}^4$$

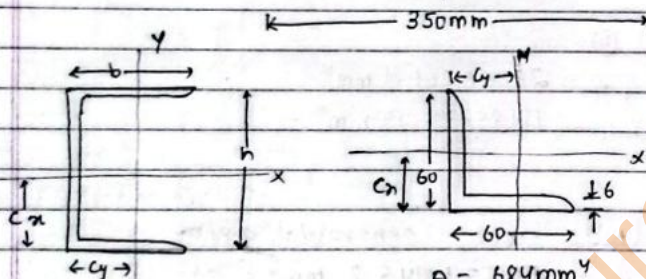
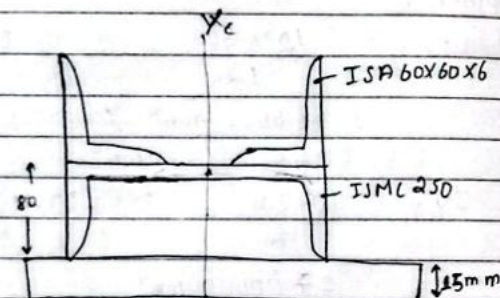
$$I_{y_c} = 85871678.18 \text{ mm}^4$$

For given channel section (ISM 225)

$A = 83.01 \text{ cm}^2$
 $h = 225 \text{ mm}$
 $b = 80 \text{ mm}$
 $c_y = 20.3 \text{ mm}$
 $I_{xx} = 2694.6 \text{ cm}^4$
 $I_{yy} = 187.2 \text{ cm}^4$



Q6)



$A = 3867 \text{ mm}^2$
 $h = 250 \text{ mm}$
 $b = 80 \text{ mm}$
 $c_y = 20.3 \text{ mm}$
 $I_{xx} = 3816.8 \text{ cm}^4$
 $I_{yy} = 219.1 \text{ cm}^4$

$A = 684 \text{ mm}^2$
 $c_x = c_y = 1.69 \text{ cm}$
 $I_x = I_y = 22.6 \text{ cm}^4$

Sol:-

Step ① locate Centroid:-

Component	y_i	A_i
Plate	7.5	5250
Channel	$15 + 80 - 20.3$ $= 74.7$	3867
Angle (Right)	$15 + 80 + 16.9$ $= 111.9$	684
Angle (Left)	111.9	684
		ΣA_i $= 10485$

$\bar{Y} = \frac{\Sigma y_i A_i}{\Sigma A_i} = \frac{481319.1}{10485} = 45.9 \text{ mm}$

Step ② MOI

(a) Plate:

$(I_{xx})_1 = \frac{15^3 \times 350}{12} + 5250(7.5 - 45.9)^2$
 $= 7839877.5 \text{ mm}^4$

$(I_{yy})_1 = \frac{15 \times 350^3}{12} + 5250(0 - 0)$
 $= 53593750 \text{ mm}^4$

(b) channel:

$(I_{xx})_2 = 3816.8 \times 10^4 + 3867(74.7 - 45.9)^2$
 $= 41375444.48 \text{ mm}^4$
 $= 5398444.48 \text{ mm}^4$

$(I_{yy})_2 = 219.1 \times 10^4 + 3867(0 - 0)^2$
 $= 2191000 \text{ mm}^4$
 $+ 3867(0 - 0)^2$
 $= 38168000 \text{ mm}^4$

© Angle section (Right & left)

$$(I_{x_c x_c})_3 = 22.6 \times 10^4 + 684(111.9 - 45.9)^2$$

$$= 9205504 \text{ mm}^4$$

$$(I_{y_c y_c})_3 = 22.6 \times 10^4 + 684\left(\frac{250}{2} - 16.9\right)^2$$

$$= 8218957.24$$

Thus,

Total Moment of inertia (MOI)

$$I_{x_c x_c} = 19649329.98 \text{ mm}^4$$

$$I_{y_c y_c} = 108199664.5 \text{ mm}^4$$

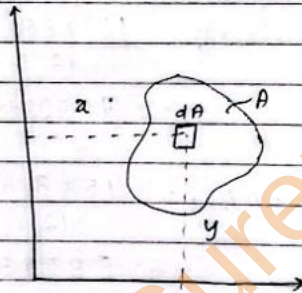
Product of Inertia:-

→ Product of inertia with respect to co-ordinate axis is the product of any elementary area with their both co-ordinates (x, y)

i.e. At a distance of ' x ' and ' y ' from y -axis and x -axis respectively.

→ Product of inertia of an elementary area dA with respect to co-ordinate axis is given by

$$dI_{xy} = xy dA$$



For whole area (A),

$$I_{xy} = \int xy dA$$

$$= xy \cdot A$$

- Unlike, moment of inertia (I_{xx} and I_{yy}), the product of inertia may be positive or negative, depending on the location of the area.
- Product of inertia is zero when one or both axes are axis of symmetry.

Parallel axis Theorem:-

From fig: $x' = a + x$
 $y' = b + y$

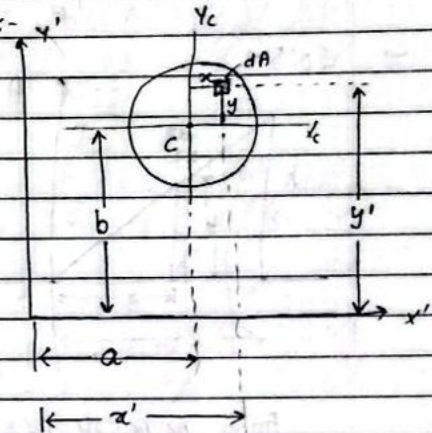


Fig:

$$I_{x'y'} = \int x'y' dA$$

$$= \int (a+x)(b+y) dA$$

$$= \int (ab + ay + bx + xy) dA$$

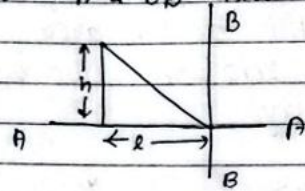
$$= \int ab dA + \int ay dA + \int bx dA + \int xy dA$$

$$\therefore I_{x'y'} = I_{x_c y_c} + abA$$

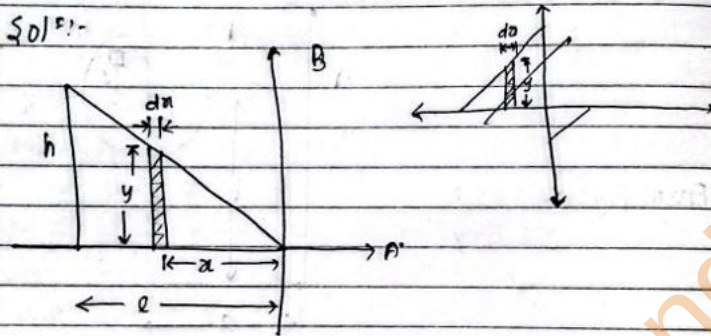
{ First moment of area about centroidal axis is zero }
 $\therefore \int ay dA = \int bx dA = 0$

$I_{x'y'} = I_{x_c y_c} + abA$
 Overall Centroid Afto Centroid
 Afto Centroid of overall centroid.
 Afto Centroid Right of overall centroid
 A → +ve → centroidal axis at side of parallel axis
 B → +ve → centroidal axis above parallel axis

(Q) Determine the product of inertia of triangle w.r.t AA & BB axes & w.r.t centroidal axes



→ Solⁿ:



From property of triangle,
 $\frac{L}{n} = \frac{h}{y} \Rightarrow y = \frac{h}{L} x$

Now,

$$I_{AB} = \int x_c y_c dA$$

$$dA = y dx$$

$$I_{AB} = \int -x \cdot \frac{y}{2} \cdot y dx$$

$$= \int_0^L -x \cdot \frac{h^2 x^2}{2L^2} dx$$

$$= -\frac{h^2}{2L^2} \left[\frac{x^4}{4} \right]_0^L$$

$$= -\frac{h^2 L^4}{8L^2}$$

$$I_{AB} = -\frac{h^2 L^2}{8}$$

We know,

From Parallel axis theorem,

$$I_{x'y'} = I_{x_c y_c} + abA$$

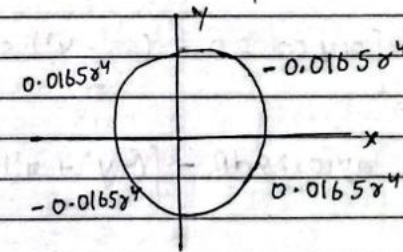
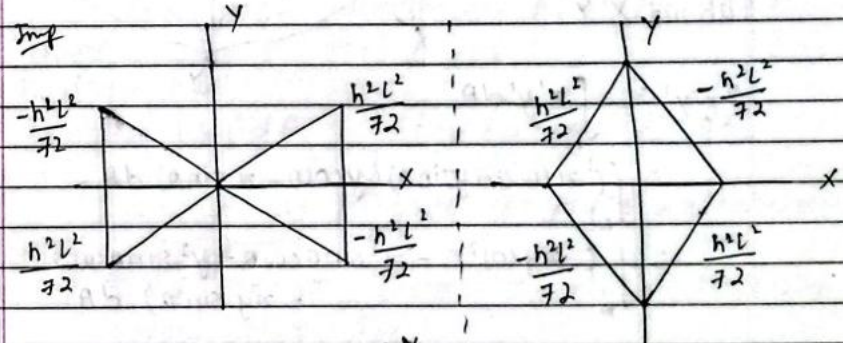
parallel centroid

$$I_{AB} = I_{x_c x_c} + \frac{1}{2} \times h \times L \times \left(-\frac{2L}{3} \right) \left(\frac{h}{3} \right)$$

$$\text{or, } -\frac{h^2 L^2}{8} = I_{x_c x_c} - \frac{h^2 L^2}{9}$$

$$\text{or, } I_{x_c x_c} = -\frac{h^2 L^2}{8} + \frac{h^2 L^2}{9} = -\frac{h^2 L^2}{72}$$

(Q) Imp



Principal Axes and Principal MOI

- If two mutually perpendicular axes about which the product of inertia is found are such that, the product of inertia becomes zero. These two axes are called principle axes.
- The moment of inertia about the principal axis is principal MOI.
- One of the principal axis yields maximum MOI while the other yields minimum MOI.

Derivation:

From Figure,

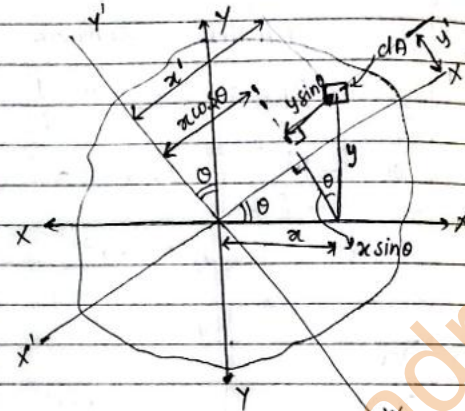
$$x' = x \cos \theta + y \sin \theta$$

$$y' = y \cos \theta - x \sin \theta$$

Now,

Product of axes about $x'y'$,

$$\begin{aligned} I_{x'y'} &= \int_A x'y' dA \\ &= \int_A (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta) dA \\ &= \int_A (xy \cos^2 \theta - x^2 \sin \theta \cos \theta + y^2 \sin \theta \cos \theta - xy \sin^2 \theta) dA \\ &= \int_A \left[xy \cos 2\theta - \frac{(x^2 - y^2) \sin 2\theta}{2} \right] dA \\ &= \int_A xy \cos 2\theta dA - \int_A \frac{(x^2 - y^2) \sin 2\theta}{2} dA \end{aligned}$$



$$I_{xx} = \int y^2 dA$$

$$I_{x'y'} = I_{xy} \cos 2\theta - \left(\frac{I_{yy} - I_{xx}}{2} \right) \sin 2\theta \quad (1)$$

The above equation (1) is general equation for finding the product of inertia about any two perpendicular axes if the MOI & product of inertia about perpendicular axes are known.

By definition, of principle axes when $I_{x'y'} = 0$, the axes $x'x'$ and $y'y'$ will become the principle axes.

Under the condition,

$$I_{xy} \cos 2\theta - \left(\frac{I_{yy} - I_{xx}}{2} \right) \sin 2\theta = 0$$

$$\text{Or, } \tan 2\theta = \frac{2 I_{xy}}{I_{yy} - I_{xx}} \quad (2)$$

This gives location of principle axes.

Now,

Principle MOI,

$$\begin{aligned} I_{x'x'} &= \int (y')^2 dA = \int (y \cos \theta - x \sin \theta)^2 dA \\ &= \int (y^2 \cos^2 \theta - 2xy \sin \theta \cos \theta + x^2 \sin^2 \theta) dA \\ &= I_{xx} \cos^2 \theta - I_{xy} \sin 2\theta + I_{yy} \sin^2 \theta \\ &= I_{xx} \left(\frac{1 + \cos 2\theta}{2} \right) - I_{xy} \sin 2\theta + I_{yy} \left(\frac{1 - \cos 2\theta}{2} \right) \\ &= \frac{I_{xx} + I_{yy}}{2} + \left(\frac{I_{xx} - I_{yy}}{2} \right) \cos 2\theta - I_{xy} \sin 2\theta \\ I_{x'x'} &= \frac{I_{xx} + I_{yy}}{2} - \left(\frac{I_{yy} - I_{xx}}{2} \right) \cos 2\theta - I_{xy} \sin 2\theta \quad (3) \end{aligned}$$

Instead of x & y we use u & v

$u \rightarrow$ Major principle axis
 $v \rightarrow$ Minor principle axis
 $I_{uu} \rightarrow$ Max I_{MOI} , $I_{vv} \rightarrow$ Min I_{MOI}

Similarly,

$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} + \left(\frac{I_{yy} - I_{xx}}{2} \right) \cos 2\theta + I_{xy} \sin 2\theta$$

Thus,

$$I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \left(\frac{I_{yy} - I_{xx}}{2} \right) \cos 2\theta + I_{xy} \sin 2\theta$$

→ det

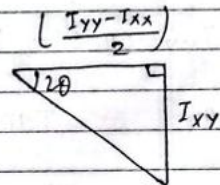
u

v

I_{uu}

I_{vv}

$$h = \sqrt{\left(\frac{I_{yy} - I_{xx}}{2} \right)^2 + I_{xy}^2}$$



Substituting the values of $\sin 2\theta$ and $\cos 2\theta$ in equation (4)

$$I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \left(\frac{I_{yy} - I_{xx}}{2} \right) \cdot \frac{\sqrt{I_{xy}^2 + \left(\frac{I_{yy} - I_{xx}}{2} \right)^2}}{\left(\frac{I_{yy} - I_{xx}}{2} \right)}$$

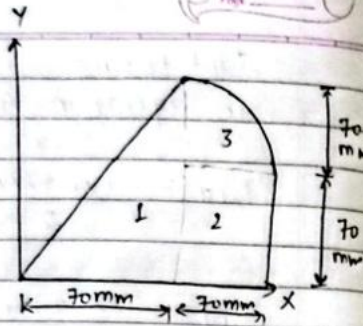
$$I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{I_{xy}^2 + \left(\frac{I_{yy} - I_{xx}}{2} \right)^2}$$

Similarly,

- (Q) Find PMOI & locate principle axes
→ Soln:

Step 1:

Locate the centroid:



S.N	Components	x_i	y_i	A_i
1	Triangle	$140/3$	$140/3$	4900
2	Square	105	35	4900
3	Quarter-circle	163.33 99.71	163.33 99.71	3848.45

$$\Sigma A = 13648.45$$

$$\bar{x} = \frac{\Sigma x_i A_i}{\Sigma A_i} = \frac{1126895.616}{13648.45} = 82.56 \text{ mm}$$

$$\bar{y} = \frac{\Sigma y_i A_i}{\Sigma A_i} = 57.43 \text{ mm}$$

$$I_{xx} = \frac{70 \times 140^3}{36} + 4900(57.43 - 46.67)^2 + \frac{70 \times 70^3}{12} + 4900(57.43 - 35)^2 + 0.055 \times (70)^4 + 3848.45(57.43 - 99.71)^2$$

$$= 18568945.72 \text{ mm}^4$$

$$a \rightarrow -ve \Rightarrow \{x_i - \bar{x} \rightarrow -ve\} \quad \begin{cases} a = x_i - \bar{x} \\ b = y_i - \bar{y} \end{cases} \quad \begin{matrix} \text{overall centroid} \\ \text{of centroid} \end{matrix}$$

$$I_{yy} = \frac{70^3 \times 140}{36} + 4900(82.56 - 46.67)^2 + \frac{70 \times 70^3}{12} + 4900(82.56 - 105)^2 + 0.055(70)^4 + 3848.45(82.56 - 99.71)^2$$

$$= 14566251.21 \text{ mm}^4$$

$$I_{xy} = \Sigma (I_{xy_c} + A a b)$$

$$= \left[\frac{70^2 \times 140^2}{72} + 4900(46.67 - 82.56)(46.67 - 57.43) \right] + \left[0 + 4900(105 - 82.56)(35 - 57.43) \right] + \left[-0.0165 \times 70^4 + 3848.45(99.71 - 82.56)(99.71 - 57.43) \right]$$

$$= 3226839.048 - 2466313.08 + 2393214.805$$

$$= 3153740.773 \text{ mm}^4$$

Now,

$$I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{yy} - I_{xx}}{2} \right)^2 + I_{xy}^2}$$

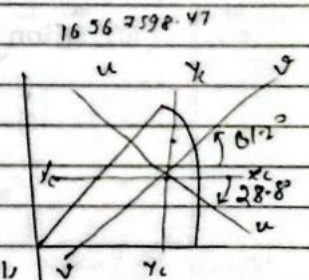
$$= 20302765.36 \text{ mm}^4$$

$$I_{vv} = 12832431.56 \text{ mm}^4$$

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}}$$

$$\theta_1 = -28.8^\circ \text{ with major axis}$$

$$\theta_2 = 61.2^\circ \text{ with minor axis}$$



Shear $\rightarrow$ slide b/w Adjacent surface.

Bearing stress $\rightarrow$ contact surface π develop $\rightarrow$

Numericals:

- Q. A steel rod of 20mm diameter and 500cm length is subjected to an axial pull of 30kN. Determine (i) the intensity of stress (ii) the strain and (iii) the elongation of rod
- $E = 200 \text{ GPa}$

$\rightarrow$ Solⁿ:-

(i) Stress (σ) = $\frac{\text{Force (P)}}{\text{Area (A)}}$

$$= \frac{30 \times 10^3}{\frac{\pi \times (20 \times 10^{-3})^2}{4}}$$

$$= 1527.88 \text{ N/m}^2 \quad 95.49 \text{ MPa}$$

E

(ii) Strain:

Longitudinal/Lateral strain (ϵ) = $\frac{\Delta L}{L} = ?$

$E = 200 \times 10^9 \text{ N/m}^2$

$E = \frac{\sigma}{\epsilon} = \frac{1527.88 \times 95.49 \times 10^6}{E}$

$\Rightarrow \epsilon = 7.63 \times 10^{-9} \quad 4.774 \times 10^{-4}$

(iii) Elongation of rod (ΔL) =

$\Delta L = \frac{PL}{AE} = 0.00238 \text{ m.}$

Indirect Stress: Stress developed in a member when it is subjected to a load which is $\perp$ to the axis of member i.e. transverse loading.

Bending stress • Torsional stress.

- (Q) A circular metal rod of 1cm diameter is loaded in tension. Where the tensile load is 8kN, the extension in the 20cm long rod is measured to be 0.029cm. Determine Young's modulus E of metal.
- $\rightarrow$ Solⁿ:-

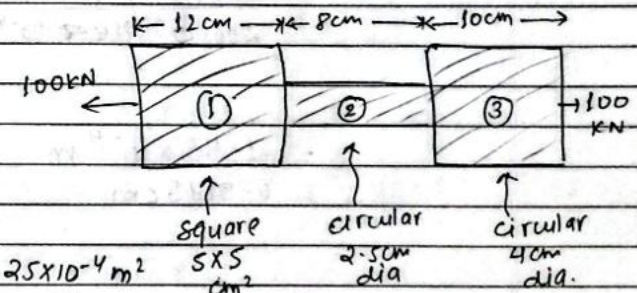
$E = \frac{\sigma}{\epsilon} = \frac{P L}{A \Delta L}$

$$= \frac{8 \times 10^3 \times 0.2}{\frac{\pi \times (0.01)^2}{4} \times (0.029 \times 10^{-2})}$$

$$= 70.25 \text{ GPa}$$

(Q) A 30cm

$\rightarrow$ Solⁿ:-

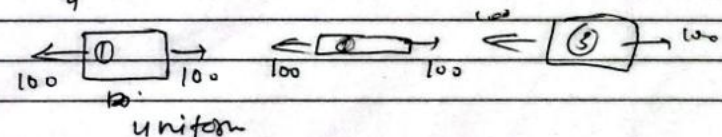


$A_1 = 25 \text{ cm}^2 = 25 \times 10^{-4} \text{ m}^2$

$A_2 = \frac{\pi (2.5 \times 10^{-2})^2}{4}$

$= 4.91 \times 10^{-4} \text{ m}^2$

$A_3 = \frac{\pi (4 \times 10^{-2})^2}{4} = 12.57 \times 10^{-4} \text{ m}^2$



$$A_1 > A_3 > A_2$$

Thus,

$$\begin{aligned} \text{Max}^{\text{m}} \text{ stress, } \sigma_{\text{max}} &= \frac{P}{A_{\text{min}}} = \frac{100 \times 10^3}{4.91 \times 10^{-4}} \\ &= 2.037 \times 10^8 \text{ Pa} \\ &= 203.7 \text{ MPa} \end{aligned}$$

$$\begin{aligned} \text{Min}^{\text{m}} \text{ stress, } \sigma_{\text{min}} &= \frac{P}{A_{\text{max}}} = \frac{100 \times 10^3}{25 \times 10^{-4}} \\ &= 4 \times 10^7 \text{ Pa} \\ &= 40 \text{ MPa} \end{aligned}$$

Total elongation, ΔL

$$= \Delta L_1 + \Delta L_2 + \Delta L_3$$

$$= \frac{P L_1}{A_1 E} + \frac{P L_2}{A_2 E} + \frac{P L_3}{A_3 E}$$

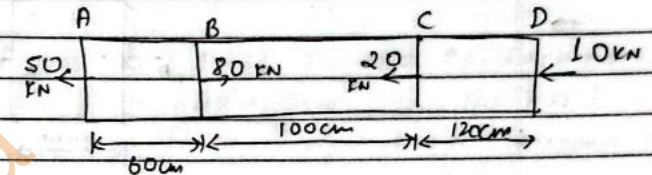
$$= \frac{100 \times 10^3}{200 \times 10^9} \left[\frac{0.12}{25 \times 10^{-4}} + \frac{0.08}{4.91 \times 10^{-4}} + \frac{0.1}{12.5 \times 10^{-4}} \right]$$

$$= 1.45 \times 10^{-4} \text{ m}$$

$$= 0.0145 \text{ cm}$$

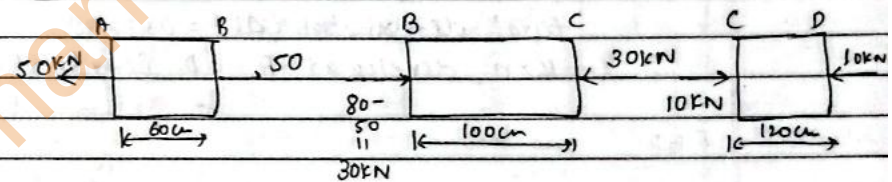
(Q) A brass bar having CSA 10 cm^2 is subjected to axial force as shown. Find total change in length of bar

$$E = 105 \text{ GPa/m}^2$$



→ Solⁿ:

$$\text{FBD } A = 10 \times 10^{-4} \text{ m}^2$$



Now,

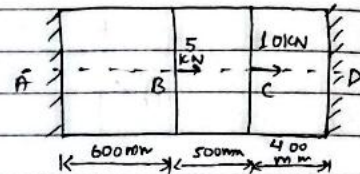
$$\Delta L = \Delta L_{AB} - \Delta L_{BC} - \Delta L_{CD}$$

$$= \frac{P_1 L_1}{A E} - \frac{P_2 L_2}{A E} - \frac{P_3 L_3}{A E}$$

$$= \frac{1 \times 10^3}{10 \times 10^{-4} \times 105 \times 10^9} [50 \times 0.6 - 80 \times 1 - 20 \times 1.2]$$

$$= -1.143 \times 10^{-4} \text{ m}$$

Q.

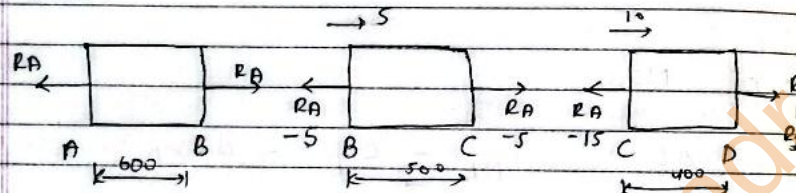


Soln:

Since fixed support in both ends
total deformation (ΔL) = 0.

& Rxn develop at A & D. (say R_A, R_D)

FBD:



$$\Delta L = 0$$

$$\text{or, } \frac{R_A \times 0.6}{AE} + \frac{(R_A - 5) \times 0.5}{AE} + \frac{(R_A - 15) \times 0.4}{AE} = 0$$

$$\text{or, } 1.5 R_A = 8.5$$

$$R_A = 5.67 \text{ kN}$$

$$R_D = R_A - 15 = -9.33 \text{ kN}$$

Stress in each part,

$$\sigma_{AB} = \frac{P_A}{A} = \frac{5.67 \times 10^3 \text{ N}}{1000 \text{ mm}^2} = 5.67 \text{ N/mm}^2 \text{ (tension)}$$

$$\sigma_{BC} = \frac{P_B}{A} = \frac{(5.67 - 5) \times 10^3}{1000} = 0.67 \text{ N/mm}^2 \text{ (tension)}$$

$$\sigma_{CD} = \frac{P_C}{A} = \frac{-9.33 \times 10^3}{1000} = -9.33 \text{ N/mm}^2 \text{ (compressive)}$$

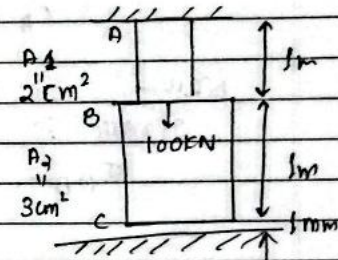
(Q) A bar ABC consists two parts AB and BC, each part being 1 m long and CSA 2 cm^2 & 3 cm^2 resp. The bar is suspended from A and there is rigid hor^l support at 2.00 m from A. A force

of 100 kN acting vertically downwards is applied at B.

Determine

(i) the Rxn (ii) stresses in part AB & BC

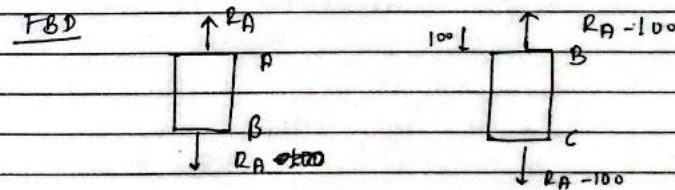
$$E = 200 \text{ GN/m}^2$$



Max: $\Delta L = 1 \text{ mm}$ if: $\Delta L > 1 \text{ mm} \rightarrow$ Rxn forces
 $\Delta L < 1 \text{ mm} \rightarrow$ NO rxn.

$$\text{Elongation } (\Delta L) \text{ at AB} = \frac{PL}{AE} = \frac{100 \times 10^3 \times 1}{2 \times 10^{-4} \times 200 \times 10^9} = 2.5 \text{ mm} > 1 \text{ mm}$$

So, it touches the ground & Rxn forces develop.



$$\Delta L = 1 \text{ mm} = 1 \times 10^{-3}$$

$$\text{or, } \frac{R_A L_A}{A_1 E} + \frac{R_B L_B}{A_2 E} = 1 \times 10^{-3}$$

$$\text{or, } \frac{R_A \times 1}{2 \times 10^{-4} \times 200 \times 10^9} + \frac{(R_A - 100) \times 1}{3 \times 10^{-4} \times 200 \times 10^9} = 1 \times 10^{-3}$$

$$\text{or, } 3 \times 10^{-4} + 2(R_A - 100) \times 10^{-4} = 6 \times 200 \times 10^9 \times 1 \times 10^{-3}$$

$$R_A = 64 \text{ kN}$$

Now,

Stresses:

$$\sigma_{AB} = \frac{R_A}{A_1} = \frac{64 \times 1000}{2 \times 10^{-4}} = 3.2 \times 10^8 \text{ N/m}^2$$

$$\sigma_{BC} = \frac{R_B}{A_2} = \frac{(64 - 100) \times 1000}{3 \times 10^{-4}} = -1.2 \times 10^8 \text{ N/m}^2$$

Generalized Hooke's Law.

When Force is applied along x-axis,

$$\epsilon_x = \frac{\sigma_x}{E}; \epsilon_y = -\frac{\nu \sigma_x}{E}; \epsilon_z = -\frac{\nu \sigma_x}{E}$$

When Force is applied along y-axis,

$$\epsilon_y = \frac{\sigma_y}{E}; \epsilon_x = -\frac{\nu \sigma_y}{E}; \epsilon_z = -\frac{\nu \sigma_y}{E}$$

When Force is applied along z-axis,

$$\epsilon_z = \frac{\sigma_z}{E}; \epsilon_x = -\frac{\nu \sigma_z}{E}; \epsilon_y = -\frac{\nu \sigma_z}{E}$$

Therefore,

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} - \frac{\nu \sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu \sigma_x}{E} - \frac{\nu \sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu \sigma_y}{E} - \frac{\nu \sigma_x}{E}$$

Poisson's ratio $\rightarrow$ Material property

$\nu = \frac{\text{Lateral strain}}{\text{Linear strain}}$

(ny) Linear strain

* Poisson's ratio, $\nu = \frac{\epsilon}{\epsilon_x}$ (0-0.5)
 $\hookrightarrow$ modulus (number)

- (a) A bar of steel is 60mm x 60mm x 180mm
 Find change in dimension of bar
 and change in volume.
 Take $E = 200 \text{ GPa}$
 & Poisson's ratio $\left(\frac{1}{m}\right) = 0.3$

$\rightarrow$ Sol:

change in volⁿ,

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$$

$$\epsilon_v = \frac{\Delta V}{V}$$

$$\Delta V = V \cdot \epsilon_v$$

Now,

Strain:-

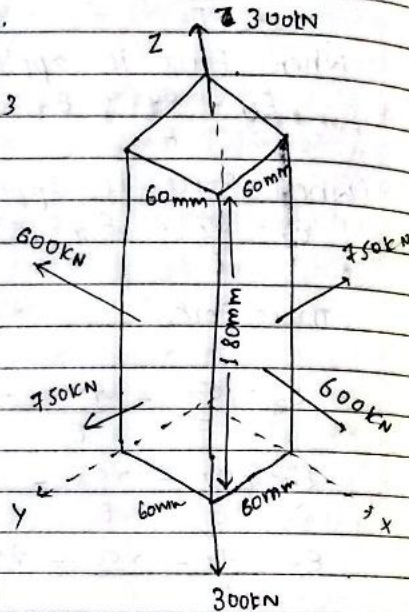
$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

Stress:-

$$\sigma_x = \frac{P_x}{A_x} = \frac{600 \times 10^3}{180 \times 60 \times 10^{-6}} = 55.56 \times 10^6 \text{ Pa}$$

$$\sigma_y = \frac{P_y}{A_y} = \frac{750 \times 10^3}{180 \times 60 \times 10^{-6}} = 69.44 \times 10^6 \text{ Pa}$$

$$\sigma_z = \frac{P_z}{A_z} = \frac{300 \times 10^3}{60 \times 60 \times 10^{-6}} = 83.33 \times 10^6 \text{ Pa}$$



If compressive $\sigma \rightarrow -ve$

Now,

Strains:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

$$= \frac{55.56 \times 10^6}{200 \times 10^9} - 0.3 \times \frac{69.44}{200 \times 10^9} - 0.3 \times \frac{83.33}{200 \times 10^9}$$

$$= 4.9 \times 10^{-5}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E}$$

$$= 1.39 \times 10^{-4}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

$$= 2.3 \times 10^{-4}$$

Now,

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$$

$$= 2.93 \times 10^{-4} + 4.18 \times 10^{-4}$$

$$\epsilon_v = \frac{\Delta V}{V}$$

$$\Rightarrow \Delta V = 2.93 \times 10^{-4} \times (60 \times 60 \times 180) = 270.864 \text{ mm}^3$$

change in dimension, along axes,

$$\Delta L_x = \epsilon_x \times L_x = 4.9 \times 10^{-5} \times 60 = 0.00294 \text{ mm}$$

$$\Delta L_y = \epsilon_y \times L_y = 1.39 \times 10^{-4} \times 60 = 0.00834 \text{ mm}$$

$$\Delta L_z = \epsilon_z \times L_z = 2.3 \times 10^{-4} \times 180 = 0.0414 \text{ mm}$$

- (Q) A short RCC column 600mm x 600mm has eight steel rods of 25mm ϕ as reinforcement. Find stresses in steel and concrete, and the elastic shortening of column if $E = 200,000$ N/mm² for steel and 10,000 N/mm² for concrete. load on column = 3000 kN and length = 3m

→ Solⁿ:

$$\phi = 25\text{mm}$$

$$E_s = 2 \times 10^5 \text{ N/mm}^2 = 2 \times 10^{11} \text{ N/m}^2$$

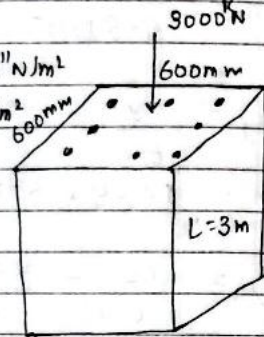
$$E_c = 1 \times 10^4 \text{ N/mm}^2 = 1 \times 10^{10} \text{ N/m}^2$$

$$\sigma_s = ?$$

$$\sigma_c = ?$$

$$\Delta L = \frac{PL}{AE}$$

$\Delta L_s = \Delta L_c$ → No cracking
→ compatibility condition.



$$\text{or, } \frac{P_s L_s}{A_s E_s} = \frac{P_c L_c}{A_c E_c}$$

$$\text{or, } \frac{P_s}{\frac{8 \times \pi \times (25 \times 10^{-3})^2}{4} \times 2 \times 10^{11}} = \frac{(3000 - P_s)}{\left(600^2 - 8 \times \pi \times (25 \times 10^{-3})^2\right) \times 1 \times 10^{10}}$$

$$\text{or, } \frac{P_s}{P_c} = 0.2205$$

$$P_s + P_c = 3000 \quad \text{Equilibrium condition}$$

$$\text{or, } 0.2205 P_c + P_c = 3000$$

$$\text{or, } P_c = 2457.88 \text{ kN}$$

$$P_s = 542.14 \text{ kN}$$

Now,

compressive stresses:

$$\text{In concrete, } \sigma_c = \frac{2457.86}{\frac{(600^2 - 8 \times \pi \times (25 \times 10^{-3})^2)}{4}} = 6.9 \text{ MN/m}^2$$

$$\text{In rod, } \sigma_s = \frac{542.14}{\frac{8 \times \pi \times (25 \times 10^{-3})^2}{4}} = 138.05 \text{ MN/m}^2$$

elastic shortening,

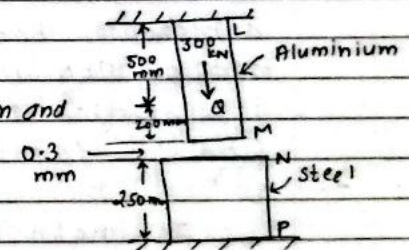
$$\Delta L = \left(\frac{PL}{AE} \right)_c = \frac{6.9 \times 10^6 \times 3}{1 \times 10^{10}} = 2.07 \text{ mm}$$

check:

$$\Delta L = \left(\frac{PL}{AE} \right)_s = \frac{138.05 \times 10^6 \times 3}{2 \times 10^{11}} = 2.07 \text{ mm}$$

- (Q) $E_{al} = 70 \text{ GPa}$ | $A_r = 2500 \text{ mm}^2$
 $A_{al} = 1250 \text{ mm}^2$ | $E_s = 210 \text{ GPa}$

Determine stress in aluminium and steel due to 300kN load applied 500mm from the ceiling.

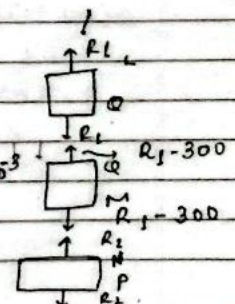


→ Solⁿ:

$$\Delta L = 0.3 \text{ mm}$$

$$\frac{R_1 \times 0.5}{1250 \times 10^{-6} \times 70 \times 10^9} + \frac{(R_1 - 300 \times 10^3) \times 0.2}{1250 \times 10^{-6} \times 210 \times 10^9} + \frac{R_2 \times 0.25}{2500 \times 10^{-6} \times 210 \times 10^9} = 0.3 \times 10^{-3}$$

$$\text{or, } 294 R_1 + 17.5 R_2 = 36225000 \quad \text{--- (1)}$$



Also, (No external forces from Q to P, so, no change in internal forces)
 Force in QN = Force in NP
 $R_1 - 300 = R_2$
 or $R_1 - 300 = R_2$
 or $R_1 - R_2 = 300 \times 1000 \text{ N}$ (1)

$$R_1 = 133.146 \text{ kN}$$

$$R_2 = -166.853 \text{ kN}$$

$$(\sigma_{PI})_{up} = \frac{R_1}{A_{PI}} = \frac{133146}{1250 \times 10^{-6}} = 106.5 \text{ MN/m}^2 \quad (\text{Tensile})$$

$$(\sigma_{PI})_{QM} = \frac{R_1 - 300 \times 1000}{1250 \times 10^{-6}} = 133.5 \text{ MN/m}^2 \quad (\text{Compressive})$$

$$(\sigma_s)_{NP} = \frac{R_2}{A_s} = \frac{166853}{2500 \times 10^{-6}} = 66.7 \text{ MN/m}^2 \quad (\text{Compressive})$$

- (Q) Two parallel steel wires 6m long, 10mm dia are hung vertically 70mm apart and support a horizontal bar at their lower ends. When a load of 9kN is attached to one of the wires, it is observed that the bar is 2.4° to horizontal. Find 'E' for wire.

→ Solⁿ:

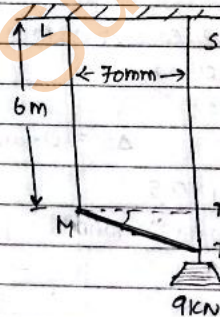
Let, Inclination of bar after application of load be θ .

Extension, $\Delta L = \delta L$

$$\delta L = 70 \tan \theta \text{ mm}$$

$$= 70 \tan (2.4^\circ)$$

$$= 2.93 \times 10^{-3} \text{ m}$$



Strain in wire,

$$\epsilon = \frac{\delta L}{L} = \frac{2.93 \times 10^{-3}}{0.6} = 4.89 \times 10^{-4}$$

Stress in wire,

$$\sigma = \frac{P}{A} = \frac{9 \times 10^3}{\pi \times (10 \times 10^{-3})^2} = 11.46 \times 10^7 \text{ N/m}^2$$

Thus,

$$E = \frac{\sigma}{\epsilon} = 2.34 \times 10^{11} \text{ N/m}^2$$

- (Q) A C-I flat, 300mm long and of 30mm X 50mm uniform section is acted upon by the following forces uniformly distributed over the respective cross section; 25kN in the dirⁿ of length (Tensile), 350kN in dirⁿ of width (Compressive) and 200kN in dirⁿ of thickness (Tensile). Determine change in vol^m of flat.

Take $E = 140 \text{ GPa}$ and $\mu = 0.4$.

→ Solⁿ:

$$\nu = \frac{1}{m} = \frac{1}{4} = 0.25$$

$$\sigma_x = \frac{P_x}{A_x} = \frac{350 \times 10^3}{30 \times 300 \times 10^{-6}} = 3.89 \times 10^7 \text{ N/m}^2 \quad (\text{compressive})$$

$$\sigma_y = \frac{P_y}{A_y} = \frac{25 \times 10^3}{50 \times 30 \times 10^{-6}} = 1.67 \times 10^7 \text{ N/m}^2 \quad (\text{Tensile})$$

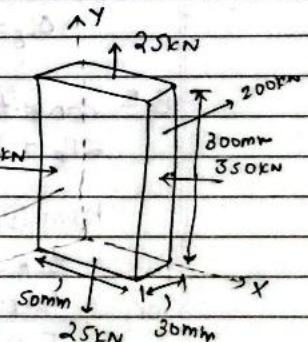
$$\sigma_z = \frac{P_z}{A_z} = \frac{200 \times 10^3}{50 \times 300 \times 10^{-6}} = 1.33 \times 10^7 \text{ N/m}^2 \quad (\text{Tensile})$$

Now, $\epsilon = \frac{\sigma}{E}$

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} = -0.33 \times 10^{-6}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E} = 1.65 \times 10^{-6}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} = 134.64 \times 10^{-6}$$



$$E_v = E_x + E_y + E_z = 631.07 \times 10^{-6} = 31.79 \times 10^{-6}$$

$$V = 300 \times 80 \times 50 = 450000 \text{ mm}^3$$

$$\Delta V = E_v \times V = 223.98 \text{ mm}^3 \quad 14.30 \text{ mm}^3$$

(decrease in volume)

Q. A rigid bar is supported by three rods in the same vertical plane and equidistant. The outer rods are of brass of length 600mm and dia 30mm. The central rod is of steel of 900mm length and dia 37.5mm. Calculate force in bars due to an applied force P if the bar remains horizontal after load is applied.

$$\text{Take } \frac{E_s}{E_b} = 2$$

→ Solⁿ:

Compatibility condition:-

$$\Delta L_B = \Delta L_S \quad \text{--- (I)}$$

Eg^m condition:-

$$2P_B + P_S = P \quad \text{--- (II)}$$

From (I),

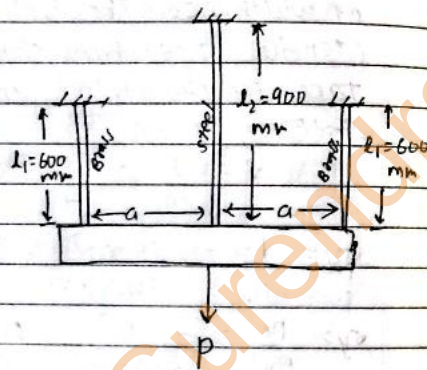
$$\Delta L_B = \Delta L_S$$

$$\text{or, } \frac{P_B L_B}{E_B A_B} = \frac{P_S L_S}{E_S A_S}$$

$$\text{or, } P_B = -0.48 P_S \quad \text{--- (III)}$$

$$P_S = 0.51 P \quad \text{--- (IV)}$$

$$P_B = 0.245 P \quad \text{--- (V)}$$



(Q) Two copper rods and one steel rod together support

→ Solⁿ:

Load together supported

by bars/rods → P

$$(\sigma_c)_{\max} = 60 \times 10^6 \text{ N/m}^2$$

$$(\sigma_s)_{\max} = 120 \times 10^6 \text{ N/m}^2$$

$$E_s = 2 E_c$$

Safe load that can be supported, P = ?

Eg^m condition,

$$P = 2 P_c + P_s$$

Compatibility condⁿ:-

$$\Delta L_S = \Delta L_C$$

$$\left(\frac{PL}{AE} \right)_S = \left(\frac{PL}{AE} \right)_C$$

$$\text{or, } \sigma_s \frac{L_S}{E_s} = \sigma_c \frac{L_C}{E_c}$$

$$\text{or, } \sigma_s = 1.5 \sigma_c \quad \text{--- (I)}$$

$$\text{For, } \sigma_c = 60 \times 10^6 \text{ N/m}^2 (\text{max})$$

$$\text{Then, corresponding stress for steel, } \sigma_s = 90 \times 10^6 \text{ N/m}^2 < 120 \times 10^6 \text{ N/m}^2$$

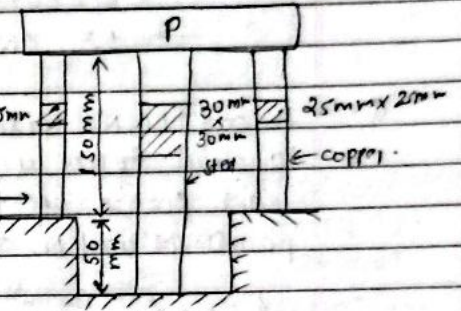
(✓)

Similarly,

$$\text{For, } \sigma_s = 120 \times 10^6 \text{ N/m}^2$$

$$\text{Then, corresponding stress for copper, } \sigma_c = 80 \times 10^6 \text{ N/m}^2 < 60 \times 10^6 \text{ N/m}^2$$

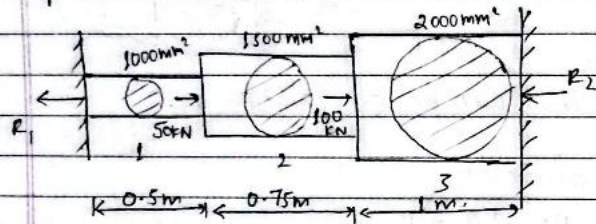
(Not safe)



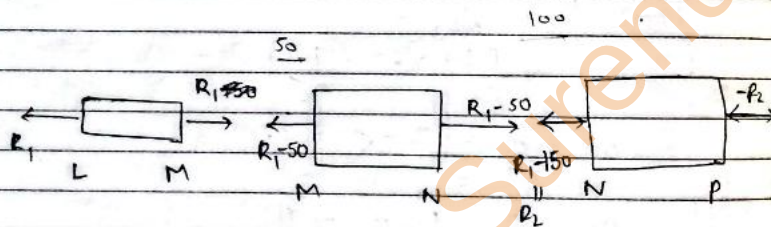
So, for safe load condition:-
 $\sigma_c = 60 \times 10^6 \text{ N/m}^2$ & $\sigma_s = 90 \times 10^6 \text{ N/m}^2$

$$\begin{aligned} P &= 2P_c + P_s \\ &= 2(\sigma_c A_c) + \sigma_s A_s \\ &= 2 \times 60 \times 10^6 \times (0.025)^2 + 90 \times 10^6 \times (0.030)^2 \\ &= 156 \text{ kN} \end{aligned}$$

Q. A bar LMNP fixed at L and P is subjected to axial forces as shown. Determine forces in each portion of bar and displacement of points M and N, take $E = 200 \text{ GPa/m}$.



So!



Compatibility condition:-

$$\Delta L = 0 \quad (\text{Net elongation zero})$$

$$\text{or, } \Delta L_1 + \Delta L_2 + \Delta L_3 = 0$$

$$\text{or, } \frac{R_1 \times 0.5}{1000 \times 10^{-6} \times E} + \frac{(R_1 - 50) \times 0.75}{1500 \times 10^{-6} \times E} + \frac{(R_1 - 150) \times 1}{2000 \times 10^{-6} \times E} = 0$$

$$\text{or, } 1.5R_1 + 1.5R_1 - 75 + 1.5R_1 - 325 = 0$$

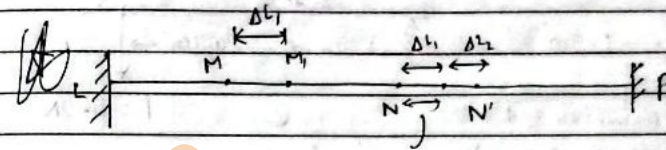
$$\Rightarrow R_1 = 66.67 \text{ kN}$$

$$-R_2 = R_1 - 150$$

$$\text{or, } -R_2 = 66.67 - 150$$

$$\text{or, } -R_2 = -83.33 \text{ kN}$$

$$R_2 = 83.33 \text{ kN}$$

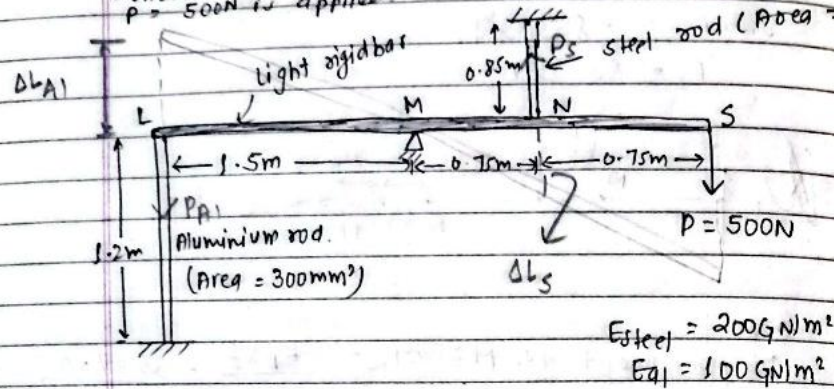


So,

$$\begin{aligned} \text{Displacement at M} &= \Delta L_1 = \frac{R_1 \times L}{(AE)} = \frac{66.67 \times 0.5 \times 1000}{1000 \times 10^{-6} \times 200 \times 10^9} \\ &= 1.667 \times 10^{-4} \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Displacement at N} &= \Delta L_1 + \Delta L_2 = 66.67 \\ &= 1.67 \times 10^{-4} + \frac{(R_1 - 50) \times 0.75}{1500 \times 10^{-6} \times 200 \times 10^9} \\ &= 1.67 \times 10^{-4} + 4.17 \times 10^{-5} \\ &= 2.087 \times 10^{-4} \text{ m} \end{aligned}$$

Q. A rigid bar LMNS pinned at M & connected to two vertical rods is shown in fig. Assuming bar was initially horizontal & rods stress free, find stress in each rod after load $P = 500\text{N}$ is applied.



Compatibility condition:
(similar Δ)

$$\frac{\Delta L_A}{\Delta L_S} = \frac{LM}{NS}$$

$$\Delta L_A = 2 \Delta L_S \quad \text{--- (I)}$$

Eq^m condition:-

$\sum M_M = 0$ (↺ +ve) (Reaction at M no need to be calculated)

$$500 \times 1.5 - P_S \times 0.75 - P_A \times 1.5 = 0$$

$$P_A = 500 - 0.5 P_S \quad \text{--- (II)}$$

From (I),

$$\Delta L_A = 2 \Delta L_S$$

$$\frac{P_A \times L_A}{A_A \times E_A} = 2 \left(\frac{P_S \times L_S}{A_S \times E_S} \right)$$

$$\frac{(500 - 0.5 P_S) \times 1.2}{300 \times 100} = \frac{2 \times P_S \times 0.85}{200 \times 200}$$

$$500 - 0.5 P_S = 1.0625 P_S$$

$$P_S = 320\text{N}$$

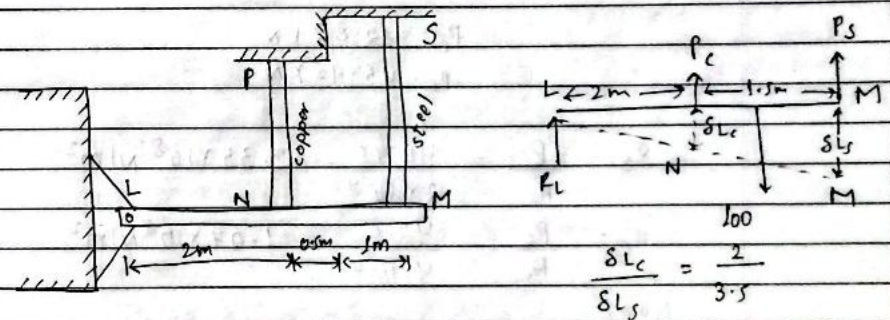
Now,

Stress in each rod,

$$\sigma_S = \frac{P_S}{A_S} = \frac{320}{200 \times 10^{-6}} = 1.6 \times 10^6 \text{ Pa}$$

$$\sigma_A = \frac{P_A}{A_A} = \frac{500 - 0.5 \times 320}{300 \times 10^{-6}} = 1.13 \times 10^6 \text{ Pa}$$

(Q)



$$\frac{\Delta L_C}{\Delta L_S} = \frac{2}{3.5}$$

$$\Delta L_C = 0.57 \Delta L_S \quad \text{--- (I)}$$

Eq^m condition:-

$$\sum M_M = 0 \quad (\curvearrowright +ve)$$

$$- P_C \times 2 - P_S \times 3.5 + 100 \times 2.5 = 0$$

$$2 P_C = 250 - 1.75 P_S \quad \text{--- (II)}$$

$$2 P_C + 3.5 P_S = 250 \quad \text{--- (IV)}$$

From (i)

$$\Delta L_c = 0.57 \Delta L_s$$

$$\text{or } \left(\frac{PL}{AE} \right)_c = 0.57 \left(\frac{PL}{AE} \right)_s$$

$$\text{or } \frac{(250 - 1.75 P_s) \cdot 0.75}{4 \times 1.2 \times 10^8} = \frac{0.57 P_s \times 1}{2 \times 2 \times 10^8}$$

$$P_c - 0.912 P_s = 0 \quad \text{--- (ii)}$$

$$\text{or } \frac{71.43}{250 - 1.75 P_s} = 0.912 P_s \quad \text{--- (iii)}$$

$$\Rightarrow P_s = 93.91 \text{ kN} \quad 26.83 \text{ kN}$$

$$P_c = 24.47 \text{ kN}$$

$$\sigma_s = \frac{P_s}{A_s} = \frac{26.83}{2 \times 10^{-4}}$$

From (ii) & (iii),

$$P_c = 42.82 \text{ kN}$$

$$P_s = 46.95 \text{ kN}$$

$$\sigma_s = \frac{P_s}{A_s} = \frac{46.95}{2 \times 10^{-4}} = 2.35 \times 10^8 \text{ N/m}^2$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{42.82}{4 \times 10^{-4}} = 1.07 \times 10^8 \text{ N/m}^2$$

Now,

$$\Delta L_s = \frac{\sigma_s L_s}{E_s} = \frac{2.35 \times 10^8 \times 1}{200 \times 10^9} = 1.175 \text{ mm}$$

$$\Delta L_c = 0.57 \times 1.175 = 0.67 \text{ mm}$$

(Q) A steel rod is 15m long is at 15°C. Find free expansion of length when temperature raised to 65°C. Find the temperature stress produced when

(a) Expansion of rod prevented.

(b) Rod is permitted to expand by 6mm

Take $\alpha = 12 \times 10^{-6}/^\circ\text{C}$

$$E = 200 \text{ GN/m}^2$$

→ Solⁿ:

$$L = 15 \text{ m}$$

$$t_1 = 15^\circ\text{C}$$

$$t_2 = 65^\circ\text{C}$$

$$\alpha = 12 \times 10^{-6}/^\circ\text{C}$$

$$E = 200 \times 10^9 \text{ N/m}^2$$

Now,

Free to expand → No stress developed

$$(i) \Delta L = \alpha L \Delta T$$

$$= 12 \times 10^{-6} \times 15 \times (65 - 15)$$

$$= 0.009 \text{ m} = 9 \text{ mm}$$

$$\epsilon = \frac{\Delta L}{L} = \frac{0.009}{15} = 0.0006$$

{ Restrict whole 9mm
so, stress
developed due
to 9mm }

$$\sigma = E \epsilon = 0.0006 \times 200 \times 10^9$$

$$= 120 \text{ MN/m}^2$$

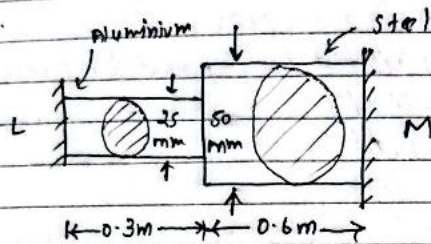
(ii) Expansion restricted = 9 - 6 = 3 mm

Thus,

$$\text{Strain } \epsilon = \frac{3}{15} = 0.2 \times 10^{-3}$$

$$\text{Stress developed due to restricted expansion } \sigma = E \epsilon = 0.2 \times 200 \times 10^9 \times 10^{-3} = 40 \text{ MN/m}^2$$

Q. A composite bar made up of aluminium and steel is held b/w two supports. The bars are stress-free at 40°C . What will be the stresses in the two bars when temp is 20°C .



$$E_s = 210 \text{ GN/m}^2$$

$$E_a = 74 \text{ GN/m}^2$$

$$\alpha_s = 11.7 \times 10^{-6} \text{ per } ^\circ\text{C}$$

$$\alpha_a = 23.4 \times 10^{-6} \text{ per } ^\circ\text{C}$$

→ Solⁿ:

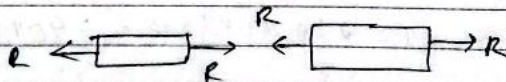
$$t_1 = 40^\circ\text{C} \text{ (stress-free)}$$

$$t_2 = 20^\circ\text{C}$$

(i) Non-yielding

Total contraction due to temp. drop

$$\begin{aligned} \Delta L &= (\alpha \Delta T L)_a + (\alpha \Delta T L)_s \\ &= (23.4 \times 10^{-6} \times 20 \times 0.3) + (11.7 \times 10^{-6} \times 20 \times 0.6) \\ &= 2.808 \times 10^{-4} \text{ m.} \end{aligned}$$



(i) If supports are non-yielding, and
(ii) supports come nearer to each other by 0.1mm. It is assumed that the change in temp. is uniform along the length of bar.

Elongation due to R

$$= \left(\frac{RL}{AE} \right)_a + \left(\frac{RL}{AE} \right)_s$$

$$= R \left[\frac{0.3}{\pi \times (25 \times 10^{-3})^2 \times 74 \times 10^9} + \frac{0.6}{\pi \times (50 \times 10^{-3})^2 \times 210 \times 10^9} \right]$$

$$= 9.71 \times 10^{-9} R$$

Thus,

$$\Delta L = 2.808 \times 10^{-4} = 9.71 \times 10^{-9} R$$

$$R = 28.91 \text{ KN}$$

Thus,

Stress developed,

$$\sigma_a = \frac{R}{A_a} = \frac{28.91}{\pi \times (25 \times 10^{-3})^2} = 58.88 \times 10^6 \text{ KN/m}^2$$

$$\sigma_s = \frac{R}{A_s} = \frac{28.91}{\pi \times (50 \times 10^{-3})^2} = 14.72 \text{ MPa}$$

(ii) If support come nearer to each other by 0.1mm

Elongation due to R

$$= \Delta L - 0.1 \times 10^{-3}$$

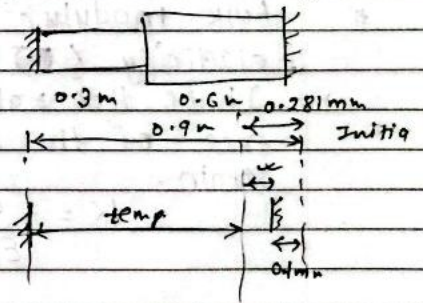
$$\text{or, } 9.71 \times 10^{-9} R = 2.808 \times 10^{-4} - 0.1 \times 10^{-3}$$

$$\Rightarrow R = 18.62 \text{ KN}$$

(Allowed contraction is,

stress developed

less than bearing

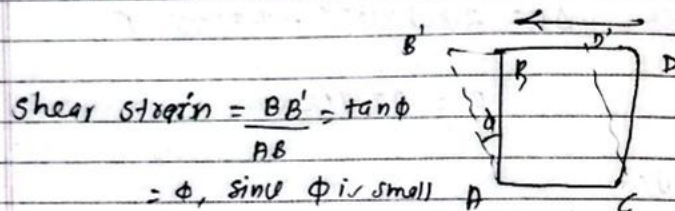


$$\sigma_{A1} = \frac{R}{A_{A1}} = 37.93 \text{ MN/m}^2$$

$$\sigma_{st} = \frac{R}{A_{st}} = 9.48 \text{ MN/m}^2$$

Shear Strain

- In case of a shearing load, a shear strain will be produced which is measured by the angle ϕ which the body distort.
- ↳ due to shear force.
 - ↳ defined by angular deformation



Modulus of Rigidity (G or C)

It is defined as the ratio of shear stress to shear strain.

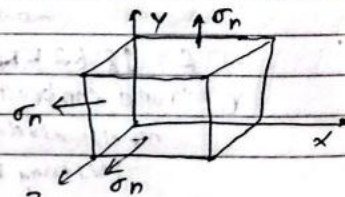
$$G = \frac{\tau}{\phi} \quad \tau \propto \phi$$

$$\tau = G \phi$$

Bulk modulus or volume modulus of elasticity (K)

- It is the ratio of normal stress on each face of the solid cube to volumetric strain.

$$K = \frac{\sigma_n}{E_v}$$



Relationship between Young's Modulus (E), Bulk Modulus (K) and Poisson's ratio (ν)

→ Solⁿ:-

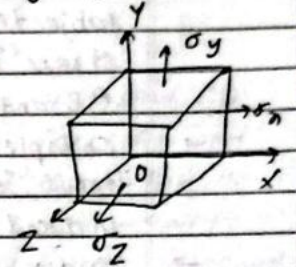
We have,

$$\text{Volumetric strain, } E_v = E_x + E_y + E_z$$

From Generalized Hooke's law,

$$E_x = \frac{\sigma_x}{E} - \frac{\nu}{E} \sigma_y - \frac{\nu}{E} \sigma_z$$

$$E_y = \frac{\sigma_y}{E} - \frac{\nu}{E} (\sigma_x + \sigma_z)$$



Thus,

$$E_v = \left[\frac{\sigma_x}{E} - \frac{\nu}{E} (\sigma_y + \sigma_z) \right] + \left[\frac{\sigma_y}{E} - \frac{\nu}{E} (\sigma_x + \sigma_z) \right]$$

$$+ \left[\frac{\sigma_z}{E} - \frac{\nu}{E} (\sigma_x + \sigma_y) \right]$$

$$= \frac{\sigma_x + \sigma_y + \sigma_z}{E} - \frac{\nu}{E} 2(\sigma_x + \sigma_y + \sigma_z)$$

$$E_v = \left(\frac{\sigma_x + \sigma_y + \sigma_z}{E} \right) (1 - 2\nu) \quad \text{--- (X)}$$

If the body is subjected to hydrostatic pressure, then $\sigma_x = \sigma_y = \sigma_z = -\sigma_n$ (i.e. compression)

Thus,

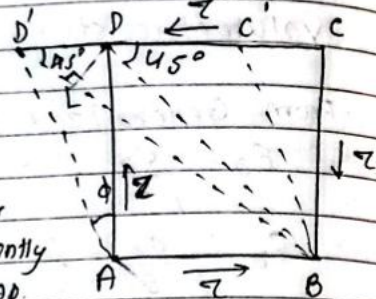
$$E_v = - \frac{3\sigma_n (1 - 2\nu)}{E}$$

$$\text{or, } E = + 3 \left(\frac{\sigma_n}{E_v} \right) (1 - 2\nu)$$

$$\text{or, } E = 3K(1 - 2\nu)$$

Relation between Young's Modulus (E), Modulus of Rigidity (G) and Poisson's Ratio (ν)

- Consider solid cube ABCD of side 'a' subjected to shear stress ' τ ' along faces AB and CD.
- Complementary shear stress ' τ ' is consequently induced in BC and AD.



- Under action of two sets of shear stress

ABCD deforms to ABC'D'

- Diagonal BD deforms to BD' → Draw DL ⊥ BD'

- Since deformation is very small, angle B'D'D may still be taken as 45°.

Now,

$$\tan \phi \approx \phi = \frac{DD'}{AD} = \frac{D'L}{\cos 45^\circ \times AD}$$

$$= \frac{D'L}{\cos 45^\circ \times (BD \cos 45^\circ)}$$

$$\phi = \frac{2D'L}{BD} \quad \text{--- (i)} \quad \left\{ \sin, \cos 45^\circ = \frac{1}{\sqrt{2}} \right\}$$

- Since BD is nearly equal to BL therefore, ϕ is increase in length of diagonal BD. Therefore, (BD ≈ BL)

Longitudinal strain in diagonal BD

$$E_L = \frac{D'L}{BD} = \frac{BD' - BD}{BD} = \frac{BD' - BL}{BD}$$

$$\text{or, } E_L = \frac{\phi \cdot D'L}{BD} = \frac{\phi}{2} \quad \left\{ \text{from (i)} \right. \\ \left. \frac{\phi}{2} = \frac{D'L}{BD} \right\}$$

We have,

$$G = \frac{\tau}{\phi} = \frac{\tau}{2E_L}$$

$$\text{or, } E_L = \frac{\tau}{2G} \quad \text{--- (ii)}$$

Due to shear stress ' τ ' on faces AB and CD and complementary shear stress on faces AD and BC, the diagonal BD is subjected to a tensile stress ' τ ' and the diagonal AC is subjected to compressive stress ' τ ' as a result of which diagonal BD elongate, and the strain is,

$$E_L = \frac{\tau}{E} = \frac{\tau}{E} (-\tau)$$

$$E_L = \frac{\tau}{E} (1 + \nu) \quad \text{--- (iii)}$$

From eqs (ii) & (iii)

$$\frac{\tau}{2G} = \frac{\tau}{E} (1 + \nu)$$

$$E = 2G(1 + \nu) \quad \text{--- (iv)}$$

$$\text{or, } G = \frac{E}{2(1 + \nu)} \quad \text{--- (v)}$$

Relation b/w E , G , K

$$E = \frac{9KG}{3K + G} \quad \text{--- (c)}$$

(Q) For the given material, Young's Modulus is 110 GN/m^2 , shear modulus is 42 GN/m^2 . Find Bulk Modulus and lateral contraction of a round bar of 37.5 mm diameter and 2.4 m length, when stretched 2.5 mm .

→ Sol:-

$$E = 110 \times 10^9 \text{ N/m}^2$$

$$G = 42 \times 10^9 \text{ N/m}^2$$

$$K = ?$$

$$E = \frac{9KG}{3K+G}$$

$$\text{or } 110 \times 10^9 = \frac{9K \times 42 \times 10^9}{3K + 42 \times 10^9}$$

$$\text{or } 330K + 4620 = 378K$$

$$\Rightarrow K = 96.25 \times 10^9 \text{ N/m}^2$$

$$K = 96.25 \text{ GN/m}^2$$

$$G = \frac{E}{2(1+\nu)}$$

$$\text{or } \nu = 0.309$$

Thus,

$$\nu = \frac{\text{Lat. strain}}{\text{Long. strain}}$$

$$\text{or } 0.309 = \frac{\text{Lat. strain}}{\frac{2.5 \times 10^{-3}}{2.4}}$$

$$\Rightarrow \text{Lat. strain} = \frac{\text{Lat. contraction}(\Delta d)}{\text{orig. dia. (d)}} = 0.00342$$

$$\hookrightarrow 37.5$$

$$\Delta d = 0.0121 \text{ mm}$$

□ Stresser on an inclined plane due to axial loading in one direction.

In eqn,
 $\Sigma F_n = 0$ (✓ +)

$\sigma_n \times AC \times 1 - \sigma_x \cos \theta \times AB \times 1 = 0$

$\sigma_n = \sigma_x \cos \theta \frac{AB}{AC}$

$\sigma_n = \sigma_x \cos^2 \theta$

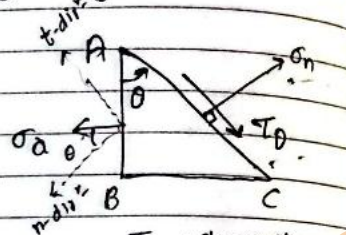
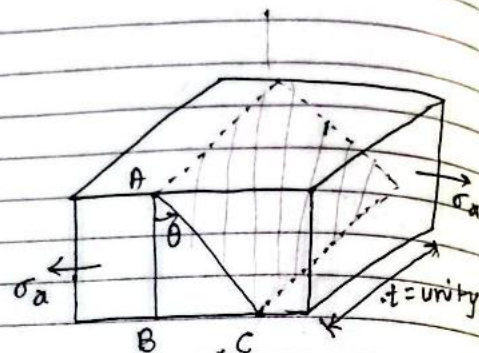
$\Sigma F_t = 0$ (✓ +ve)

$\tau_{\theta} \times AC - \sigma_x \sin \theta \times AB = 0$

$\tau_{\theta} = \sigma_x \sin \theta \frac{AB}{AC}$

$\tau_{\theta} = \frac{1}{2} \sigma_x \sin 2\theta$

$\tau_{\theta} = \frac{1}{2} \sigma_x \sin 2\theta$



$\tau_{\theta} \rightarrow$ shear stress
 (Tangential)

$\sigma_n \rightarrow$ normal stress

$\theta \rightarrow$ vertical axis
 & for Anticlock.
 (currently)

Case (I), When, $\theta = 0^\circ$

$\sigma_n = \sigma_x$

$\tau_{\theta} = 0$

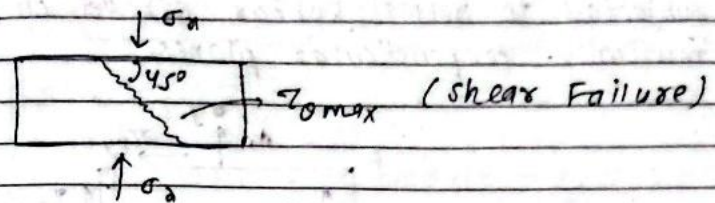
Case (II) When $\theta = 90^\circ$

$\sigma_n = 0$

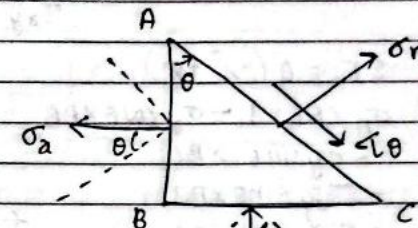
$\tau_{\theta} = 0$

case III:

τ_{θ} is maximum when $\theta = 45^\circ$



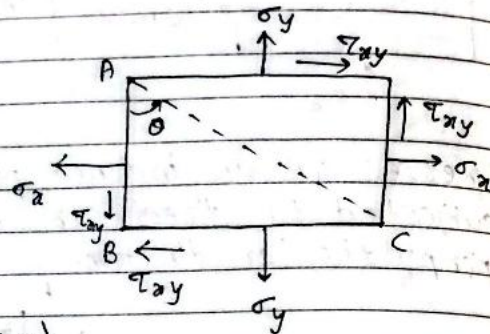
□ Stresser on an inclined plane of a body subjected to direct stresser in two mutually perpendicular direction.



$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2} \right) + \left(\frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta$

$\tau_{\theta} = \left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta$

Stresser on an inclined plane of an element subjected to normal & shear stresser on two mutually perpendicular planes.



$$\Sigma F_n = 0 \quad (\nearrow +ve)$$

$$\text{or, } \sigma_n \times AC \times 1 - \sigma_x \cos \theta \times AB - \sigma_y \sin \theta \times BC - \tau_{xy} \sin \theta \times AB - \tau_{xy} \cos \theta \times BC = 0$$

$$\text{or, } \sigma_n = \frac{\sigma_x \cos \theta \times AB}{AC} + \frac{\sigma_y \sin \theta \times BC}{AC} + \tau_{xy} \sin \theta \frac{AB}{AC} + \tau_{xy} \cos \theta \frac{BC}{AC}$$

$$\text{or, } \sigma_n = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + \tau_{xy} \sin \theta \cos \theta + \tau_{xy} \sin \theta \cos \theta$$

$$\text{or, } \sigma_n = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + \frac{1}{2} \tau_{xy} \sin 2\theta$$

$$\text{or, } \sigma_n = \sigma_x \left(\frac{1 + \cos 2\theta}{2} \right) + \sigma_y \left(\frac{1 - \cos 2\theta}{2} \right) + \tau_{xy} \sin 2\theta$$

$$\frac{\sigma_x}{2} + \frac{\sigma_y}{2} (\cos^2 \theta - \sin^2 \theta)$$

$$\text{or, } \sigma_n = \left(\frac{\sigma_x + \sigma_y}{2} \right) + \left(\frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta \quad (1)$$

Similarly,

$$\Sigma F_t = 0 \quad (\searrow +ve)$$

$$\text{or, } \tau_\theta \times AC - \sigma_x \sin \theta \times AB + \sigma_y \cos \theta \times BC - \tau_{xy} \sin \theta \times BC + \tau_{xy} \cos \theta \times AB = 0$$

$$\text{or, } \tau_\theta = \frac{\sigma_x \sin \theta \cos \theta - \sigma_y \cos \theta \sin \theta}{\sin^2 \theta + \cos^2 \theta} + \tau_{xy} \sin^2 \theta - \tau_{xy} \cos^2 \theta$$

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta \quad (2)$$

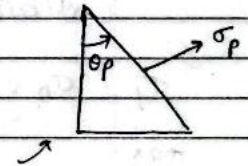
Maximum and Minimum Normal Stress

$$\frac{d\sigma_n}{d\theta} = 0$$

$$\Rightarrow \left\{ \left(\frac{\sigma_x - \sigma_y}{2} \right) \cdot 2 \sin 2\theta \right\} + 2\tau_{xy} \cos 2\theta = 0$$

$$\text{or, } \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\text{or, } \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (3)$$



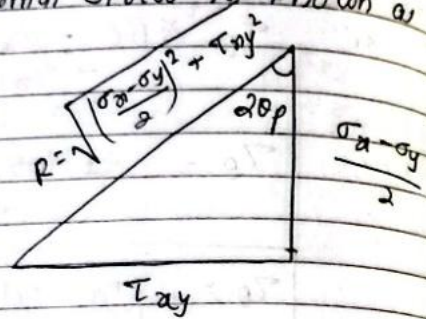
Equation (3) gives the orientation of principle plane.

Equating $\tau_\theta = 0$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Principle plane:

A plane in which only normal stress of maximum value is present and no shear stress acts is known as principle plane and the maximum normal stress is known as principle stress.



Now,

$$\sin 2\theta = \frac{\tau_{xy}}{\sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}}$$

$$\cos 2\theta = \frac{(\frac{\sigma_x - \sigma_y}{2})}{\sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}}$$

Substituting in eq: (1) we get,

$$\sigma_1 = \sigma_n = \frac{\sigma_x + \sigma_y}{2} + \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2} \quad (4)$$

(Principle stress)

$$\sigma_2 = \sigma_n = \frac{\sigma_x + \sigma_y}{2} - \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2} \quad (5)$$

Maximum and Minimum shear stress

$$\frac{d\tau_\theta}{d\theta} = 0$$

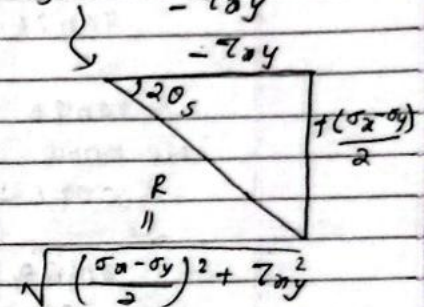
$$\text{on } 2 \cos 2\theta \cdot (\frac{\sigma_x - \sigma_y}{2}) + 2 \sin 2\theta \tau_{xy} = 0$$

$$\text{on } \tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{+ (\frac{\sigma_x - \sigma_y}{2})}{-\tau_{xy}} \quad (6)$$

From 4,

$$\cos 2\theta = \frac{-\tau_{xy}}{R}$$

$$\sin 2\theta = \frac{+(\frac{\sigma_x - \sigma_y}{2})}{R}$$



Substituting in eq: (2), we get

$$\tau_{max} = \frac{+(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}{R}$$

$$= \frac{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}{\sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}}$$

$$\tau_{max} = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2} \quad (7)$$

$$\therefore \tau_{max} = \frac{\sigma_1 - \sigma_2}{2} \quad (8)$$

{ From (4) & (5) }

→ Clockwise → +ve
 → Anti clock → -ve
 τ_{xy} → Horizontal surface of element
 stress on shear

Thus,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

gives orientation

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Here,

$$\tan 2\theta_s = -1$$

$$\tan 2\theta_p$$

$$\tan 2\theta_s = -\cot 2\theta_p$$

We know,

$$\cot \theta = -\tan(\theta + 90^\circ)$$

So,

$$\tan 2\theta_s = \tan(2\theta_p + 90^\circ)$$

$$2\theta_s = 2\theta_p + 90^\circ$$

$$\theta_s = \theta_p + 45^\circ \quad \text{--- (9)}$$

Q. Determine principle stresses, Maximum shear stress. Also determine stress component on the element obtained by rotating the element 30° anticlockwise.

→ Solⁿ:

$$\tau_{xy} = -48 \text{ MPa}$$

$$60 \text{ MPa} = \sigma_y$$

$$48 \text{ MPa}$$

$$(y\text{-face})$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{100 + 60}{2} + \sqrt{\left(\frac{100 - 60}{2}\right)^2 + (-48)^2}$$

$$= 132 \text{ MPa}$$

Every angle in Mohr's circle is double angle.

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 28 \text{ MPa}$$

$$2\theta_{p1} = \tan^{-1} \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right)$$

100 60

$$\theta_{p1} = -33.7^\circ$$

$$\theta_{p2} = \theta_{p1} + 90^\circ = -33.7^\circ + 90^\circ = 56.3^\circ$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{132 - 28}{2} = 52 \text{ MPa}$$

$$\theta_{s1} = \theta_{p1} + 45^\circ = -33.7^\circ + 45^\circ = 11.3^\circ$$

$$\theta_{s2} = \theta_{s1} + 90^\circ = 11.3^\circ + 90^\circ = 101.3^\circ$$

(b) If element is rotated by 30° anticlock.

$\theta = 30^\circ$ (anticlock)

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta + \tau_{xy} \sin 2\theta$$

30° 30°

$$= 48.43^\circ \text{ MPa}$$

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta - \tau_{xy} \cos 2\theta$$

$$= 41.32^\circ \text{ MPa}$$

$z(\text{MPa})$

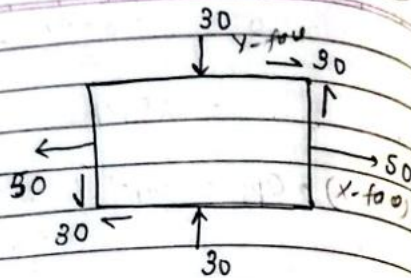


(Q2)

→ Solⁿ:

$$\sigma_x = 50 \text{ MPa}, \sigma_y = -30 \text{ MPa}$$

$$\tau_{xy} = 30 \text{ MPa}$$



$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 60 \text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= -40 \text{ MPa}$$

$$\tan 2\theta_p = \frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\theta_{p1} = 18.43^\circ$$

$$\theta_{p2} = 18.43^\circ + 90^\circ = 108.43^\circ$$

$$\theta_{s1} = 18.43^\circ + 45^\circ = 63.43^\circ$$

$$\theta_{s2} = \theta_{s1} + 90^\circ = 153.43^\circ$$

$$\text{On } \theta_{s1} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 50 \text{ MPa}$$

For $\theta = 30^\circ$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= 56 \text{ MPa}$$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - \tau_{xy} \cos 2\theta$$

$$= 19.64 \text{ MPa.}$$

Q(3) Direct stress of 120 MN/m^2 in tension and 90 MN/m^2 in compression are applied to an elastic material at rt. angles to each other. If maximum principle stress is not to exceed 150 MPa in tension, to what shearing stress can the material be subjected? What is the max^m resulting shear stress? Also find the magnitude of other principle stress and its inclination to 120 MPa stress.

→ Solⁿ:

Maximum principle stress, (σ_1)
 $= 150 \text{ MPa}$

 $\tau_{xy} = ?$ $\sigma_x = 120 \text{ MPa}$ $\sigma_y = -90 \text{ MPa}$

Then,

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{or, } 150 = \frac{120 - 90}{2} + \sqrt{\left(\frac{120 + 90}{2}\right)^2 + \tau_{xy}^2}$$

∴ $\tau_{xy} = 84.85 \text{ MPa.}$] Shearing stress

Maximum resulting shear stress,

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{150 + 120}{2} = 135 \text{ MPa.}$$

Other principle stress, $\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$
 $= -120 \text{ MPa.}$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\Rightarrow \theta_{p1} = 19.47^\circ \quad \text{angle made by } \sigma_1 \text{ with vertical axis (y)}$$

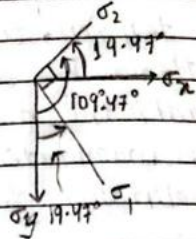
$$\theta_{p2} = \theta_{p1} + 90^\circ = 109.47^\circ$$

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\Rightarrow \theta_{s1} = -28.92^\circ - 25.53^\circ$$

$$\theta_{s2} = \theta_{s1} + 90^\circ = 61.08^\circ - 64.47^\circ$$

→ Inclination of σ_2 with $\sigma_x = 19.47^\circ$



Q(4) At a point in a material under stress, the intensity of the resultant stress is on a certain plane is 50 MPa (Tension), inclined at 30° to normal of that plane. Find the resultant stress on second plane, find principle stress and orientation. Find the plane of maximum shear and intensity.

Horizontal plane

$$\sigma_y = 30, \sigma_x = 43.3$$

$$\tau_{xy} = 25$$

$$\textcircled{1} R = \sqrt{30^2 + 25^2}$$

$$\sigma_R = \sqrt{\sigma_y^2 + \tau_{xy}^2} = 39.051 \text{ MPa}$$

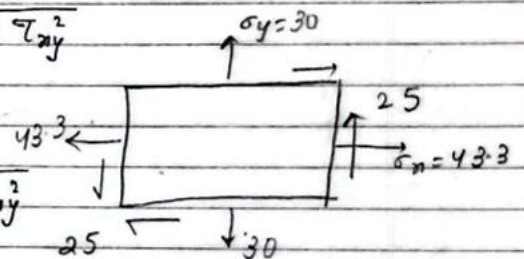
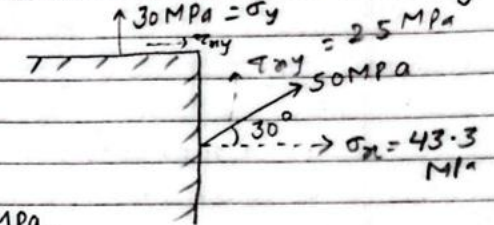
Principal stresses:

$$\textcircled{II} \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{max} = 62.52 \text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{min} = 10.78 \text{ MPa}$$



Orientation: $\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$

$$\theta_{p1} = 37.55^\circ$$

$$\theta_{p2} = 37.55^\circ + 90^\circ = 127.55^\circ$$

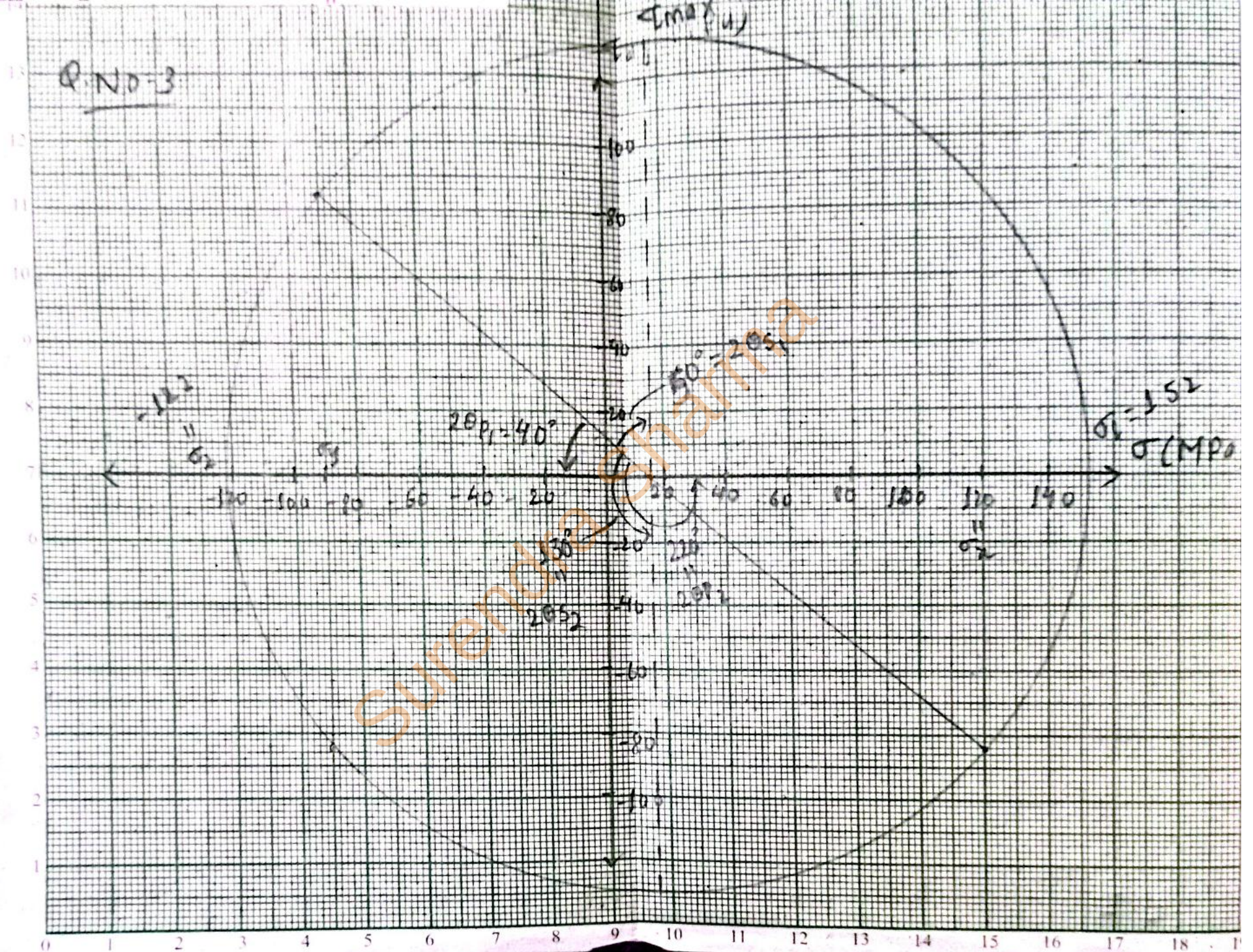
$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} \quad \{ \theta_s = \theta_p + 45^\circ \}$$

$$\Rightarrow \theta_{s1} = 27.85^\circ - 7.45^\circ$$

$$\theta_{s2} = \theta_{s1} + 90^\circ = 82.55^\circ$$

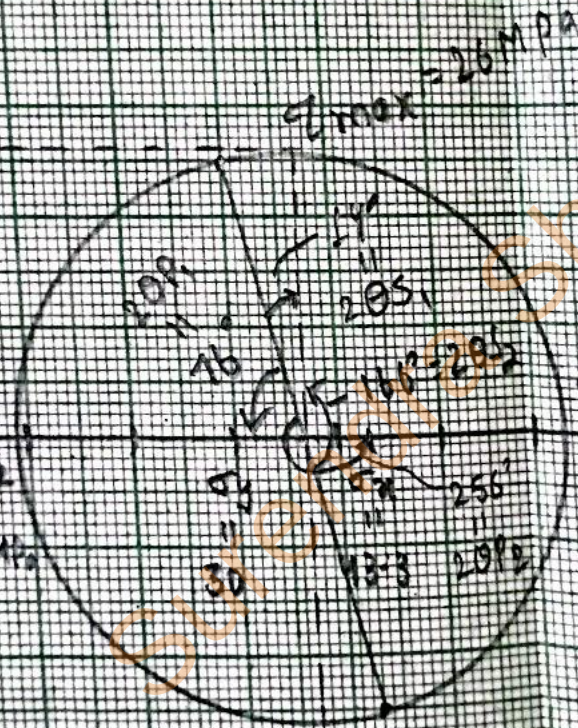
$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 25.87 \text{ MPa}$$

Q.No-3



$\tau_{xy}(\text{MPa})$

Q-NO-4



MPa.

→ Thickness of vessel less than $\frac{1}{20}$ of internal dia (d), the stress distribution over thickness of wall is nearly equal.

$$\rightarrow t < \frac{d}{20}$$

→ Stress distribution is assumed uniform over the thickness of cylinder wall.

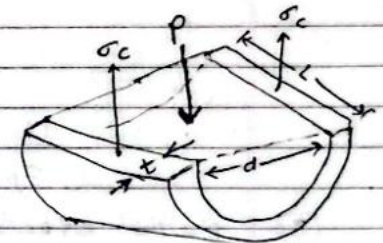
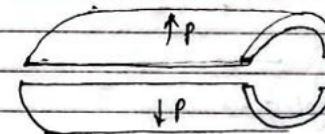
→ Used for storing fluids (liquid or gas) under pressures

Types of Stress

1. Hoop or circumferential stress

→ act in a tangential direction to circumference of shell

→ denoted by σ_c .



P → stress/pressure

$$\text{Bursting force} = \text{Resisting strength}$$

$$P \times (d \times l) = \sigma_c (2 \times t \times l)$$

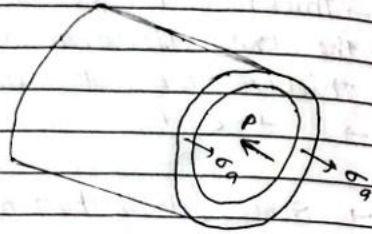
or,

$$\sigma_c = \frac{Pd}{2t} \quad \text{--- (1) (Normal Stress)}$$

∝ Bursting

(2) Longitudinal Stress.

act parallel to longitudinal axis of shell
 $\rightarrow \sigma_l \rightarrow$ longitudinal stress



Pressure exerted by fluid on end plate is equal to resisting force developed in material along longitudinal dir.

$$P \times \frac{\pi d^2}{4} = \sigma_l \times \pi d t$$

$$\sigma_l = \frac{Pd}{4t} \quad \text{--- (2) (Normal stress)}$$

(1) & (2)

$$2\sigma_l = \sigma_c \quad \text{--- (3)}$$

(3) Radial stress $\rightarrow$ Negligible.

$$(1) \sigma_c = \frac{Pd}{2t}$$

$$(2) \sigma_l = \frac{Pd}{4t}$$

$$(3) \sigma_c > \sigma_l \Rightarrow \text{Principle stress}$$

σ_c & σ_l are principle stress

Thus,

Maximum shear stress,

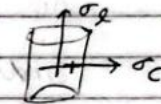
$$\tau_{max} = \frac{\sigma_c - \sigma_l}{2} \left(\frac{\sigma_1 - \sigma_2}{2} \right)$$

$$= \frac{Pd}{2t} - \frac{Pd}{4t}$$

$$\tau_{max} = \frac{Pd}{8t} \quad \text{--- (4)}$$

Strains

(i) Circumferential/Hoop strain



$$\epsilon_c = \frac{\sigma_c}{E} - \nu \frac{\sigma_l}{E} \quad (\text{Generalized Hooke's law})$$

↑ due to circumferential hoop stress (σ_c) circumferential strain due to σ_l

$$= \frac{Pd}{2t} - \nu \frac{Pd}{4t}$$

$$\epsilon_c = \frac{Pd}{4tE} (2 - \nu) \quad \text{--- (5)}$$

(ii) Longitudinal strain

$$\epsilon_l = \frac{\sigma_l}{E} - \nu \frac{\sigma_c}{E}$$

$$= \frac{Pd}{4t} - \nu \frac{Pd}{2t}$$

$$\epsilon_l = \frac{Pd}{4Et} (1 - 2\nu) \quad \text{--- (6)}$$

③ Volumetric strain
Circumference of cylinder, $C = \pi d$ — (a)
or, $\frac{C}{d} = \pi$

$$\text{or, } \frac{\delta C}{\delta d} = \pi$$

$$\text{or, } \delta C = \pi \delta d \text{ — (b)}$$

Dividing (b) by (a)

$$\frac{\delta C}{C} = \frac{\pi \delta d}{\pi d}$$

$$\text{or, } \epsilon_c = \frac{\delta d}{d} \text{ — (7)}$$

Now,

$$\text{Longitudinal strain } \epsilon_l = \frac{\delta l}{l}$$

$$\text{Volume of cylinder, } V = \frac{\pi d^2}{4} \times l$$

Change in volume,

$$\delta V = \frac{\pi (d + \delta d)^2}{4} \times (l + \delta l) - \frac{\pi d^2 l}{4}$$

$$\delta V = \frac{\pi (d^2 + 2d\delta d)(l + \delta l)}{4} - \frac{\pi d^2 l}{4}$$

$$= \frac{\pi (d^2 \delta l + 2d \cdot l \cdot \delta d)}{4} \quad \left\{ \begin{array}{l} \text{Neglecting} \\ \text{small quantities} \\ \delta d^2, \delta l \cdot \delta d \end{array} \right.$$

$$= \frac{\pi d^2 l}{4} \left(\frac{\delta l}{l} + \frac{2\delta d}{d} \right)$$

$$\delta V = V(\epsilon_l + 2\epsilon_c)$$

$$\epsilon_v = \frac{\delta V}{V} = \epsilon_l + 2\epsilon_c$$

$$\epsilon_v = \frac{Pd}{4Et} (1 - 2\nu + 4 - 2\nu)$$

$$\epsilon_v = \frac{Pd}{4tE} (5 - 4\nu) \text{ — (8)}$$

Spherical shells:

$$\text{Bursting force} = \text{Resisting force} \quad \left\{ \begin{array}{l} d = \text{Diameter of Spherical shell} \\ t = \text{Thickness of shell} \\ p = \text{Intensity of pressure} \\ \sigma_c = \text{Circumferential (or hoop) stress produced} \end{array} \right.$$

$$p \times \frac{\pi}{4} d^2 = \pi d t \cdot \sigma_c$$

$$\text{or, } \sigma_c = \frac{Pd}{4t} \text{ — (9)}$$

Now,

$$\sigma_l = \sigma_c = \frac{Pd}{4t} \text{ — (10)}$$

$$\epsilon_c = \frac{\sigma_c}{E} - \nu \frac{\sigma_c}{E} \quad \left\{ \text{Both dir. m. } \sigma_c \text{ (or } \sigma_l) \right.$$

$$= \frac{Pd}{4tE} - \nu \frac{Pd}{4tE}$$

$$\epsilon_c = \frac{Pd}{4tE} (1 - \nu) \text{ — (11)}$$

Volumetric strain, $\epsilon_v = \epsilon_L + 2\epsilon_c$ ($\epsilon_L + \epsilon_c$)
 $= 3\epsilon_c$

$$\epsilon_v = \frac{3Pd}{4tE} (1 - \nu) \quad \text{--- (1)}$$

$$\epsilon = \frac{Pd}{4t\eta_f} ; \sigma_t = \frac{Pd}{4t\eta_c}$$

dia $\rightarrow$ If said external, it's ext.
 not mentioned, int. dia.

(dia only given $\rightarrow$ sphere)
 ~ dia. & length given $\rightarrow$ cylinder

Q.1 Calculate the increase in volⁿ of boiler shell 5m long and 1.8m in diameter when subjected to an internal pressure of 4 N/mm^2 . The wall thickness is such that maximum tensile stress is not to exceed 40 N/mm^2 . Take $E = 2.1 \times 10^5 \text{ N/mm}^2$, poisson's ratio = 0.3.

$\rightarrow$ Solⁿ:

$$E = 2.1 \times 10^5 \text{ N/mm}^2$$

$$\text{Int. dia, } d = 1.8 \text{ m}$$

$$l = 5 \text{ m}$$

$$p = 4 \times 10^6 \text{ N/m}^2$$

$$\delta V = ?$$

$$\sigma_c = 40 \times 10^6 \text{ N/m}^2$$

$$\sigma_c = \frac{Pd}{2t}$$

$$\Rightarrow t = \frac{4 \times 10^6 \times 1.8}{2 \times 40 \times 10^6} = 0.09 \text{ m} = 90 \text{ mm}$$

$$\epsilon_v = \frac{Pd}{4tE} (5 - 4\nu)$$

$$= \frac{4 \times 10^6 \times 1.8}{4 \times 0.09 \times 2.1 \times 10^{11}} (5 - 4 \times 0.3)$$

$$\epsilon_v = \frac{\delta V}{V} = 0.000362$$

$$\Rightarrow \delta V = 0.000362 \times \frac{\pi \times 1.8^2}{4} \times 5$$

$$= 0.0046047$$

$$= 4.6047 \times 10^{-4} \text{ m}^3$$

$$= 4.6047 \times 10^{-3} \text{ m}^3$$

$$= 4.6047 \times 10^6 \text{ mm}^3$$

- ② A cylindrical shell is 1m diameter and 6m long. It is subjected to 40 kg/cm^2 internal pressure. If the max. principle stress is not to exceed 1500 kg/cm^2 and poisson's ratio $= 0.25$. Find max^m intensity of shear stress developed and also find change in volume. Assume $E = 2 \times 10^6 \text{ kg/cm}^2$.
- Solⁿ.

$$d = 1\text{m}$$

$$l = 6\text{m}$$

$$\sigma_c = 1500 \text{ kg/cm}^2 \quad P = 40 \text{ kg/cm}^2$$

$$\nu = 0.25$$

$$E = 2 \times 10^6 \text{ kg/cm}^2$$

$$\tau_{\text{max}} = ?$$

$$\delta V = ?$$

$$\sigma_c = \frac{Pd}{2t}$$

$$\Rightarrow t = \frac{40 \times 100}{2 \times 1500} = 0.0133\text{m} = 1.33\text{cm}$$

$$\tau_{\text{max}} = \frac{Pd}{8t}$$

$$= \frac{40 \times 100}{8 \times 1.33}$$

$$= 375.94 \text{ kg/cm}^2$$

$$\delta V = E_v \times V$$

$$= \frac{Pd}{4tE} (1 - \nu^2) \times \frac{\pi}{4} \times 100^2 \times 600$$

$$= \frac{40 \times 100}{4 \times 1.33 \times 2 \times 10^6} (1 - 0.25^2) \times \frac{\pi}{4} \times 100^2 \times 600$$

$$= 7086.29 \text{ cm}^3$$

- ③ A copper plate vessel in shape of a thin sp. shell 50cm radius and 1cm thickness is completely filled with a fluid at atm. pressure. Additional fluid is then pumped till the pressure increase by 10 MN/m^2 . Find vol^m of this additional fluid. Given that poisson's ratio $= 0.26$ and $E = 100 \text{ GN/m}^2$ for shell material.
- Solⁿ.

$$E = 100 \times 10^9 \text{ N/m}^2$$

$$d = 1\text{m}$$

$$t = 0.01\text{m}$$

$$P = 10 \times 10^6 \text{ N/m}^2$$

$$\nu = 0.26$$

$$E_v = \frac{3Pd}{4tE} (1 - \nu^2)$$

$$= 0.00940.00555$$

$$\delta V = 0.00555 \times \frac{4}{3} \times \pi \times (0.5)^3$$

$$= 2905.8 \text{ cm}^3$$

- ⑥ A cylinder vessel whose ends are closed is made of steel plate 3mm thick, the internal length and dia are 50cm and 25cm respectively. Determine longitudinal and circumferential stresses in cylindrical shell due to int. fluid pressure of 3 MN/m^2 . Also, calculate increase in length, dia. and vol^m of vessel. Take $E = 200 \text{ GPa}$, $\nu = 0.3$.

→ Sol:

ends closed $\Rightarrow$ longitudinal stress developed

$$t = 3\text{mm} = 3 \times 10^{-3}\text{m}$$

$$d = 50\text{cm} = 0.5\text{m}$$

$$d = 25\text{cm} = 0.25\text{m}$$

$$P = 3 \times 10^6 \text{ N/m}^2$$

$$E = 2 \times 10^{11} \text{ N/m}^2$$

$$\nu = 0.3$$

$$\sigma_L = \frac{Pd}{4t} = \frac{6.25 \times 10^7 \text{ N/m}^2}{4t} = 62.5 \text{ MPa}$$

$$\sigma_c = \frac{Pd}{2t} = 2\sigma_L = 12.5 \times 10^7 \text{ N/m}^2 = 125 \text{ MPa}$$

$$\delta l = \epsilon_L \cdot L$$

$$= \left(\frac{\sigma_L}{E} - \nu \frac{\sigma_c}{E} \right) \cdot L$$

$$= 6.25 \times 10^{-5} \text{ m}$$

$$\delta d = \epsilon_c \cdot d$$

$$= \left(\frac{\sigma_c}{E} - \nu \frac{\sigma_L}{E} \right) d = \frac{1 \times 10^6}{2 \times 10^{11}} (12.5 - 0.3 \times 62.5) \times 25$$

$$= 0.01328 \text{ cm}$$

$$\delta V = \epsilon_v \cdot V$$

$$= (2\epsilon_c + \epsilon_L) V$$

$$= \frac{Pd}{4tE} (5 - 4\nu) \times V$$

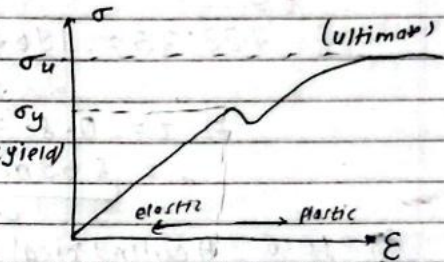
$$= \frac{3 \times 10^6 \times 0.25 \times (5 - 4 \times 0.3) \times \pi \times 0.25^2 \times 0.5}{4 \times 3 \times 10^{11} \times 2 \times 10^{11}}$$

$$= 29.14 \times 10^{-6} \text{ m}^3 = 29.14 \text{ cm}^3$$

(7) A cylindrical water tank having vertical axis open at the top. Its inside diameter is 500cm and height is 20cm. The tank is full of water and is made of structural steel and having yield stress of 2200 kg/cm². Determine thickness of steel sheet used if efficiency of longitudinal joint is 80% and factor of safety 3.

→ Sol:

Allowable = $\frac{\text{Yield stress}}{\text{Factor of Safety}}$



Factor of Safety $\left\{ \begin{array}{l} \text{Load} \times \text{FOS} \\ \text{Material strength} \end{array} \right\}$ FOS: 1.5-3 for buildings

Load $\rightarrow$ unpredictable $\therefore$ FOS required. (uncertainty)

Here,

Yield stress, $\sigma_y = 2200 \text{ kg/cm}^2$

For safe side $\left\{ \begin{array}{l} \text{Load: over estimate } \therefore \text{Load} \times \text{FOS} \\ \text{Material strength: under estimate, } \therefore \frac{\text{Load}}{\text{FOS}} \end{array} \right.$

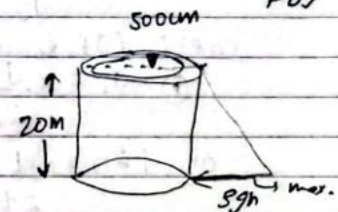
Yield stress = Max. stress

$$\sigma_c = 2200 \text{ kg/cm}^2$$

$$\sigma_L = 0 \text{ (cause open)}$$

$$\text{Allowable stress} = \frac{\text{Yield}}{\text{FOS}}$$

$$= \frac{2200}{3} = 733.33 \text{ kg/cm}^2$$



As top of tank only hoop circumferential stress
 longitudinal stress is max at bottom
 The pressure of water inside tank is max at bottom
 $\sigma_c = 733.33 \text{ kg/cm}^2$

$$\sigma_c = \frac{Pd}{2t\eta_l} \quad \left\{ \begin{array}{l} P = \rho gh \\ = 1000 \times 9.81 \times 20 \\ = 196200 \text{ N/m}^2 \end{array} \right.$$

$$733.33 = \frac{196200 \times 0.500 \times 10^{-4}}{2 \times t \times 0.8 \times 9.81}$$

$t =$

$$\therefore 733.33 = \frac{196200 \times 500 \times 10^{-4}}{2 \times t \times 0.8 \times 9.81 \times 10^4}$$

$$\Rightarrow t = 0.853 \text{ cm}$$

$$= 8.53 \text{ mm.}$$

Eg: 10.6 A cylindrical air drum 2.25m in diameter with plates 1.2cm thick. The efficiencies of the longitudinal and circumferential joints are respectively 75% and 40%. If the tensile stress in the plating is to be limited to 120 MN/m^2 , find max safe air pressure.

→ Soln:

$$d = 2.25 \text{ m}$$

$$t = 1.2 \text{ cm}$$

$$\eta_l = 0.75, \eta_c = 0.4$$

$$(P_{\text{safe}})_{\text{air}} = ?$$

Case-I (If σ_c considered max.)

$$\textcircled{1} \sigma_c = \frac{Pd}{2t\eta_c}$$

$$\text{or, } 120 = \frac{Pd}{2t\eta_c}$$

$$\text{or, } P = 0.96 \text{ MPa}$$

Efficiencies included
 So, (σ_c) might not be greater than σ_l
 So, both need to be calculated

change in diameter, $\delta d = E_c \times d$

$$E_v = E_l + 2E_c$$

'or' Case-II (If σ_l considered max.)

$$\sigma_l = \frac{Pd}{4t\eta_l}$$

$$\text{or, } 120 = \frac{P \times 2.25}{4 \times 1.2 \times 10^{-2} \times 0.75}$$

$$\Rightarrow P = 1.024 \text{ MPa.}$$

Safe pressure = 0.96 MPa
 (Min of two)

For, $\sigma_c = 120 \text{ MPa}$
 If $P = 0.96$
 $\sigma_l = 112.5 \text{ MPa}$
 $< 120 \text{ MPa}$
 So, safe press.

Esay, for $P = 1.024 \text{ MPa}$,

$$\sigma_l = 120 \text{ MPa}, \sigma_c = 12 \text{ MPa}$$

($> 120 \text{ MPa}$)
 Not okay

Eg: 10.9 A built up cylindrical shell of 300mm dia, 3m long and 6mm thick is subjected to an internal pressure of 2 MN/m^2 . Calculate the change in length, diameter and volume of cylinder under the pressure if efficiencies of longitudinal and circumferential joints are 80% & 50% respectively.

$$E = 200 \text{ GNM}^2, \delta l = E_l \cdot L$$

$$m = 3.5$$

$$\left(\frac{\sigma_l}{E} - \frac{\nu \sigma_c}{E} \right)$$

Can't use dir. formula $\frac{Pd}{4t}$

(Since it is derived w/o considering efficiency)

$$d = 300 \text{ mm} = 0.3 \text{ m}$$

$$l = 3 \text{ m}$$

$$t = 6 \times 10^{-3} \text{ m}$$

$$p = 2 \text{ MN/m}^2$$

$$\eta_l = 0.8 ; \eta_c = 0.5, E = 200 \text{ GN/m}^2$$

$$\nu = \frac{1}{m} = \frac{1}{3.5} = 0.2857$$

Now,

$$\text{Circumferential strain, } \epsilon_c = \frac{\sigma_c}{E} - \nu \frac{\sigma_l}{E} \quad (1)$$

$$\text{Longitudinal strain, } \epsilon_l = \frac{\sigma_l}{E} - \nu \frac{\sigma_c}{E} \quad (2)$$

$$\text{Volumetric strain, } \epsilon_v = \epsilon_l + 2\epsilon_c \quad (3)$$

Thus,

$$\sigma_c = \frac{pd}{2t\eta_l} = \frac{2 \times 10^6 \times 0.3}{2 \times 6 \times 10^{-3} \times 0.8} = 62.5 \text{ MN/m}^2$$

$$\sigma_l = \frac{pd}{4t\eta_c} = \frac{2 \times 10^6 \times 0.3}{4 \times 6 \times 10^{-3} \times 0.5} = 50 \text{ MN/m}^2$$

From (1),

$$\epsilon_c = \frac{1 \times 10^6}{200 \times 10^9} [62.5 - 0.2857 \times 50]$$

$$= 0.00024$$

$$\text{From (2), } \epsilon_l = \frac{1 \times 10^6}{200 \times 10^9} [50 - 0.2857 \times 62.5]$$

$$= 0.0001607$$

$$\epsilon_v = \epsilon_l + 2\epsilon_c = 0.0006407$$

Thus,

$$\delta d = d \epsilon_c = 7.2 \times 10^{-6} \text{ m (Increase)}$$

$$\delta l = l \epsilon_l = 482.1 \times 10^{-6} \text{ m (Increase)}$$

$$\delta v = V \epsilon_v = \left(\frac{\pi \times 0.3^2 \times 3}{4} \right) \times 0.0006407$$

$$= 135.86 \text{ cm}^3 \text{ (Increase)}$$

Eg: 10.10 A copper cylinder 90cm long, 40cm external dia and wall thickness of 6mm has its both ends closed by rigid blank flanges. It is initially full of oil at atmospheric pressure. Calculate additional volume of oil which must be pumped into it in order to raise the oil pressure 5MN/m² above atm. pressure. For copper, assume, $E = 100 \text{ GN/m}^2$ and poisson's ratio = $\frac{1}{3}$. Take bulk modulus of oil as 2.6 GN/m^2 .

→ Solⁿ:

$$t = 6 \text{ mm} = 6 \times 10^{-3} \text{ m}$$

$$l = 90 \text{ cm} = 0.9 \text{ m} ; D = 40 \text{ cm}$$

$$d = 40 - 2 \times 0.6 = 38.8 \text{ cm} = 0.388 \text{ m}$$

$$E = 1 \times 10^{11} \text{ N/m}^2$$

$$K = 5 \times 10^6 \text{ N/m}^2$$

$$\nu = \frac{1}{3} = 0.333$$

$$K = \frac{\sigma_n}{\epsilon_v}$$

$$\delta v_1 = \epsilon_v \cdot V$$

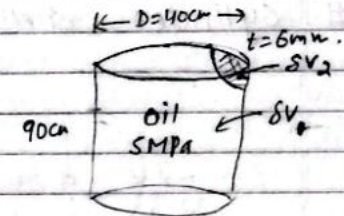
$$= \frac{pd}{4tE} (5 - \nu) \times \frac{\pi d^2}{4} \times l$$

$$= \frac{5 \times 10^6 \times 0.388 (5 - \frac{1}{3})}{4 \times 6 \times 10^{-3} \times 1 \times 10^{11}}$$

$$\times \pi \times 0.388^2 \times 0.9$$

$$= 0.000315 \text{ m}^3$$

$$= 315.4 \text{ cm}^3$$



δv_1 : Increase press. s.t. previously kept oil gets compressed & gap of δv_2 will remain in closed cylinder.
 δv_2 : Extra liquid that can be added.

$$V = \text{oil} + \delta v_1 + \delta v_2$$

$$K = \frac{\sigma_n}{\epsilon_v}$$

$$\text{or } 2.6 \text{ GPa} = \frac{5 \text{ MPa}}{\frac{\delta V_2}{V}}$$

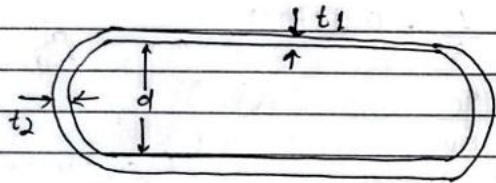
$$\text{or } \delta V_2 = \frac{5}{2.6 \times 10^3} \times \frac{\pi \times 0.388^2}{4} \times 0.9$$

$$\delta V_2 = 204.64 \text{ cm}^3$$

Thus,

$$\begin{aligned} \text{Total additional } \delta V &= \delta V_1 + \delta V_2 \\ \text{vol} &= 315.4 + 204.64 \\ &= 520.04 \text{ cm}^3 \end{aligned}$$

Cylinder shell with hemispherical ends



t_1 = thickness of cylinder

t_2 = Thickness of hemisphere ends

For cylindrical portion,

$$\sigma_{c1} = \frac{Pd}{2t_1}; \sigma_{h1} = \frac{Pd}{4t_1}$$

$$\epsilon_{c1} = \frac{Pd}{4t_1 E} (2-\nu); \epsilon_{h1} = \frac{Pd}{4t_1 E} (1-2\nu)$$

For hemisphere ends,

$$\sigma_{c2} = \frac{Pd}{4t_2}; \epsilon_{c2} = \frac{Pd}{4t_2 E} (1-\nu)$$

case ①: For no distortion at junction

$$\epsilon_{c1} = \epsilon_{c2} \quad \left\{ \begin{array}{l} \delta c_1 = \delta c_2 \\ \epsilon_{c1} \times d = \epsilon_{c2} \times d \\ \epsilon_{c1} = \epsilon_{c2} \end{array} \right.$$

$$\therefore \frac{t_2}{t_1} = \frac{1-\nu}{2-\nu}$$

$$\Rightarrow t_2 < t_1 \quad \text{--- (*)}$$

Case ②: For maximum utilization of material

$$\sigma_{c1} = \sigma_{c2}$$

$$\text{or } \frac{t_2}{t_1} = \frac{1}{2} = 0.5 \quad \text{--- (**)}$$

(9) A thin cylindrical shell is made of steel plate, has hemispherical end of diameter 300mm and wall thickness 2mm. Determine the thickness of cylindrical portion if there is no distortion on the junction under pressure. Take $E = 200 \text{ GPa}$, $\nu = 0.3$.

→ Soln:

$$d = 300 \text{ mm}$$

$$t_2 = 2 \text{ mm (hemisphere)}$$

For no distortion condition,

$$\frac{t_2}{t_1} = \frac{1-\nu}{2-\nu}$$

$$\begin{aligned} \text{or } t_1 &= \frac{2(2-0.3)}{(1-0.3)} \\ &= 4.86 \text{ mm} \end{aligned}$$

Q A boiler shell is to be made of 15mm thick plate having tensile stress of 120 MN/m^2 . If efficiencies of longitudinal & circumferential joints are 70% and 30% respectively, determine
(a) Max. permissible diameter for internal pressure of 2 MN/m^2

(b) Permissible intensity of internal pressure when shell diameter is 1.5m.

→ Solⁿ:

∵ No length given $\Rightarrow$ spherical shell

$$t = 15 \text{ mm} = 15 \times 10^{-3} \text{ m}$$

$$\eta_l = 70\%$$

$$\eta_c = 30\%$$

$$P = 2 \times 10^6 \text{ N/m}^2$$

(a) When $\sigma_l = \sigma_c = 120 \text{ MN/m}^2$.

$$\text{For } P = 2 \text{ MN/m}^2$$

$$\sigma_c = \frac{Pd}{4t\eta_c}$$

$$\Rightarrow \text{or, } 120 = \frac{2 \times d}{4 \times 15 \times 10^{-3} \times 0.3}$$

$$d = 2.5 \text{ m.}$$

$$\sigma_l = \frac{Pd}{4t\eta_l}$$

$$\text{or, } 120 = \frac{2 \times d}{4 \times 15 \times 10^{-3} \times 0.7}$$

$$d = 1.08 \text{ m}$$

Thus,

Max. permissible diameter is 1.08m
(Min of two)

(b)

When, $d = 1.5 \text{ m}$

$$\text{Case I: } \sigma_l = 120 \text{ MN/m}^2$$

$$\text{or, } 120 = \frac{Pd}{4t\eta_l}$$

$$\text{or, } \frac{120 \times 4 \times 15 \times 10^{-3} \times 0.7}{1.5} = P$$

$$\Rightarrow P = 3.36 \text{ MN/m}^2$$

$$\sigma_c = 120 \text{ MN/m}^2$$

$$\text{or, } 120 = \frac{Pd}{4t\eta_c}$$

$$\text{or, } P = \frac{120 \times 4 \times 15 \times 10^{-3} \times 0.3}{1.5}$$

$$P = 1.44 \text{ MN/m}^2$$

permissible intensity of internal pressure
= 1.44 MN/m^2
(Min of 2)

→ Torsion is the twisting moment or couple or torque which tends to rotate the plane perpendicular to longitudinal axis.

Assumptions

1. The material of shaft is uniform all over its length.
2. The shaft, circular in shape section, remains circular after loading.
3. Sections that are plane before twisting remain so after twisting.
4. The twist along the length of shaft is uniform all over its length.
5. The stresses are within the elastic limits i.e. shear stress is proportional to shear strain.
6. Distance b/w any two normal pts remain same after application of torque.

Derivation of Torsional Formula

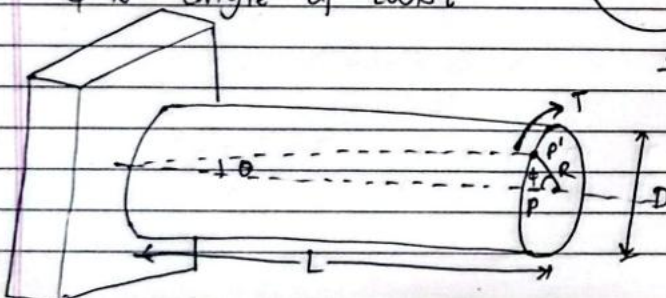
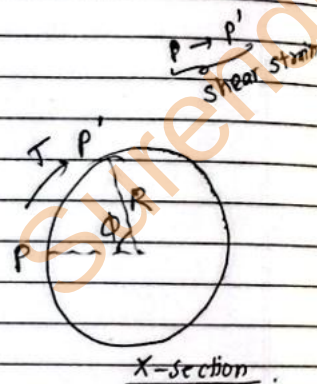
Let, T is equal to maximum twisting torque.

D is diameter of shaft

L is length of shaft

θ is shear strain

ϕ is angle of twist



From figure,

$$PP' = \phi R$$

$$\text{and } \tan \theta = \frac{PP'}{L}$$

For small rotation

$$\theta_{\max} = \frac{PP'}{L} = \frac{\phi R}{L} \quad \dots (1)$$

Within elastic limit, according to Hooke's law,

$$\tau_{\max} = G\theta \quad \dots (2) \quad \{\tau \propto \theta\}$$

here, G is Modulus of Rigidity

$\tau_{\max}$ is shear stress on outermost surface of shaft.

From (1) & (2), $\{\tau_{\min} = 0 \text{ (at centre)}\}$

$$\tau_{\max} = G \frac{\phi R}{L}$$

$$\frac{\tau_{\max}}{R} = \frac{G\phi}{L} \quad \dots (3)$$

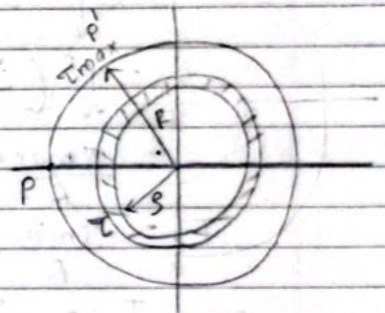
Consider radial distance r and shear stress τ , here variation is linear. (obeys Hooke's law)

From similar triangles,

$$\frac{\tau_{\max}}{R} = \frac{\tau}{r} \quad \dots (4)$$

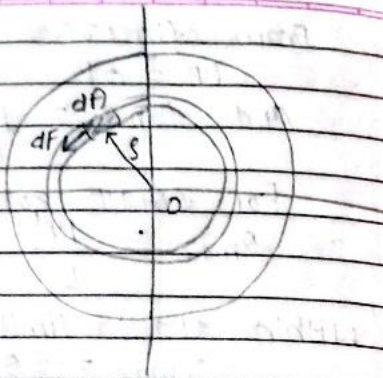
From (3) & (4)

$$\frac{\tau_{\max}}{R} = \frac{\tau}{r} = \frac{G\phi}{L} \quad \dots (5)$$



$T_p \rightarrow$ Resulting moment (Resulting capacity)

$\rightarrow$ Consider an elemental area dA at a distance s , then
Resulting force at this area is dF



$$dF = \tau dA$$

Again,

Torque of this force about the centre,
 $dT = dF \times s$ (Resulting moment to the externally applied torque)
 $= \tau dA s$

Total torque can be obtained by integrating from 0 to R which must be equal to external torque,

$$\begin{aligned} T &= \int_0^R dT \\ &= \int_0^R \tau s dA \\ &= \int_0^R \left(\frac{\tau_{max}}{R} s \right) s dA \quad [\text{From (4)}] \\ &= \frac{\tau_{max}}{R} \int_0^R s^2 dA \end{aligned}$$

$$T = \frac{\tau_{max}}{R} \cdot I_p$$

$$\text{or, } \frac{T}{I_p} = \frac{\tau_{max}}{R} \quad \text{--- (6)}$$

Here, $I_p = J$ (Polar moment of inertia)
 $= \int_0^R s^2 dA$

From (5) and (6),

$$\frac{\tau_{max}}{R} = \frac{T}{I_p} = \frac{4\phi}{L} = \frac{T}{I_p} \quad \text{--- (7)}$$

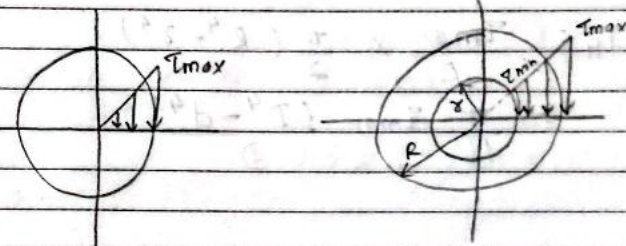
This is known as TORSIONAL EQUATION.

(a) For solid shaft

$\rightarrow$ Shear stress is zero at centre and max. at outer surface.

(b) For hollow shaft

$\rightarrow$ Shear stress is minimum at inner surface and maximum at outer surface.



Solid

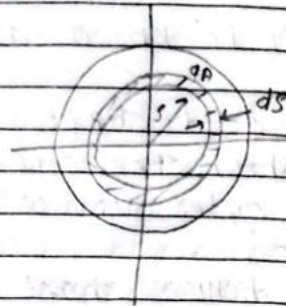
□ Torsional resistance developed by solid & Hollow shaft

(a) Solid shaft

$$\begin{aligned} I_p &= \int_0^R s^2 dA \\ &= \int_0^R s^2 \cdot 2\pi s ds \\ &= 2\pi \int_0^R s^3 ds \end{aligned}$$

$$\left\{ I_p = \frac{2\pi R^4}{4} = \frac{\pi R^4}{2} \right\} \rightarrow \text{Solid shaft.}$$

$$\begin{aligned} T_s &= \frac{T_{\max} \cdot I_p}{R} \\ &= \frac{T_{\max} \cdot \pi R^4}{R \cdot 2} \\ &= \frac{\pi T_{\max} D^3}{16} \end{aligned}$$



(b) Hollow Shaft

$$I_p = \frac{\pi}{2} (R^4 - r^4) \quad \left\{ I_p = 2\pi \int_r^R s^3 ds \right\}$$

$$\begin{aligned} T_H &= \frac{T_{\max}}{R} \times \frac{\pi}{2} (R^4 - r^4) \\ &= \frac{\pi T_{\max} (D^4 - d^4)}{16 D} \end{aligned}$$

□ Power transmitted by Shaft

Suppose a Force of 'F' N is acting tangentially to a shaft of radius 'R', then torque T,

$$T = FR \text{ (Nm)}$$

If due to this torque the shaft rotates by ω rad/sec, then,

$$\text{Work done/sec} = T\omega \text{ (Nm/sec)}$$

$$= T(2\pi f)$$

$$= 2\pi f T \text{ Watt}$$

∴ Power transmitted,

$$\left\{ P = \frac{2\pi f T}{1000 \times 60} \text{ kW} \right\}$$

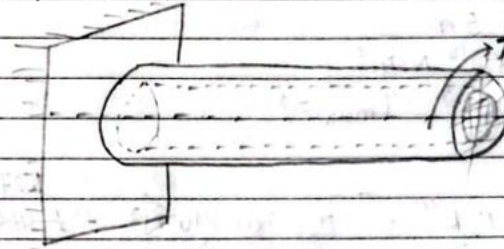
where, $f = \text{rev per min.}$

□ Composite Shaft

(A) Series shaft



(B) Shaft in parallel.

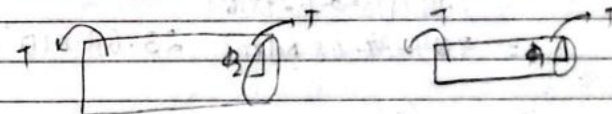


(A) ~~Series~~ Series Shaft

→ Each shaft transmit the same torque, T.

→ Angle of twist ϕ ,

$$\phi = \phi_1 + \phi_2 + \phi_3 + \dots$$



$$T = \text{same}$$

$$\phi = \phi_1 + \phi_2 + \dots$$

- (B) Shaft in parallel
 $T = T_1 + T_2 + T_3 + \dots$
 $\phi_1 = \phi_2 = \phi_3 = \phi_4$

Numericals

- (1) For a shaft of 60mm diameter, determine the maximum shear stress developed if twisting moment of 10kNm is applied on it. Also, calculate the angle between of twist in a 1.5m length of shaft. If the material is steel, take $G = 90 \text{ GPa}$. Assume material is loaded upto elastic limit only.

→ Solⁿ:-

$$D = 60 \times 10^{-3} \text{ m}$$

$$T = 10 \text{ kNm}$$

$$L = 1.5 \text{ m}$$

$$G = 90 \times 10^9 \text{ N/m}^2$$

$$\tau_{\max} = ? \quad \phi_{\max} = ?$$

Here,

$$\text{For shaft, } I_p = \frac{\pi R^4}{2} = \frac{\pi (60 \times 10^{-3})^4}{2} = 1.27 \times 10^{-6} \text{ m}^4$$

$$\tau_{\max} = \frac{T}{I_p} \times \left(\frac{60 \times 10^{-3}}{2} \right)$$

$$= \frac{10 \times 10^3}{1.27 \times 10^{-6}} \times (30 \times 10^{-3})$$

$$= 724637.68 \text{ N/m}^2 \quad 23.62 \times 10^7 \text{ N/m}^2$$

$$\frac{T}{I_p} = \frac{G\phi}{L}$$

$$\phi = \frac{10 \times 10^3 \times 1.5}{1.27 \times 10^{-6} \times 90 \times 10^9}$$

$$= 0.13 \text{ rad}$$

- (3) A hollow shaft is to transmit 200kW at 80 rpm. If the max. shear stress is not to exceed 60MPa and internal diameter is 0.6 of external diameter, find diameter of shaft.

→ Solⁿ:-

$$P = 200 \text{ kW}$$

$$f = 80 \text{ rpm}$$

$$\tau_{\max} = 60 \text{ MPa}$$

$$d = 0.6 D$$

For hollow shaft

$$T_H = \frac{\pi}{16} \tau_{\max} \frac{(D^4 - d^4)}{D}$$

$$P = \frac{2\pi f T}{60} \text{ kW}$$

$$1000 \times 60$$

$$\Rightarrow T = \frac{2387324.15}{1000} \text{ Nm} = 2.387 \times 10^6 \text{ Nm}$$

So,

$$2.387 \times 10^6 = \frac{\pi}{16} \times 60 \times 10^6 \left[\frac{D^4 - (0.6D)^4}{D} \right]$$

$$D = 0.0637 \text{ m} = 63.7 \text{ mm}$$

$$d = 0.6D = 38.22 \text{ mm}$$

$$D = 137.23 \text{ mm} \quad 137.53 \text{ mm}$$

$$d = 0.6D = 82.34 \text{ mm} \quad 79.52 \text{ mm}$$

(5) A solid circular shaft transmits 75 kW power at 200 rpm. Calculate the shaft diameter if the twist in the shaft is not to exceed 1 degree in 2 m length of shaft and shear stress is limited to 50 MPa. Take $G = 100 \text{ GPa}$.

→ Solⁿ:-

$$P = 75 \text{ kW}$$

$$f = 200 \text{ rpm}$$

$$\phi = D = ?$$

$$\phi \leq 1^\circ = \frac{\pi}{180} \text{ rad.}$$

Two conditions cannot be satisfied

$$\tau_{\max} = 50 \text{ MPa}$$

Case I:-

$$\text{For } \phi = 1^\circ = \frac{\pi}{180} \text{ rad}$$

$$\frac{\tau}{I_p} = \frac{G \phi}{L}$$

$$I_p = \frac{\pi R^4}{2}$$

$$\tau = \frac{\pi \tau_{\max} R^3}{2}$$

$$\frac{\pi}{2} \times 50 \times 10^6 \times R^3 = \frac{1 \times 10^\circ \times \pi}{2 \times 180}$$

$$\Rightarrow R =$$

Case I: $\phi = 1^\circ$

$$P = \frac{2\pi f T}{1000 \times 60}$$

$$\Rightarrow T = 3580.986 \text{ Nm}$$

Now,

$$\frac{T}{I_p} = \frac{G \phi}{L}$$

$$\text{or, } \frac{3580.986}{\frac{\pi}{2} \times R^4} = \frac{1 \times 10^\circ \times \pi}{2 \times 180}$$

$$\Rightarrow R = 0.0402 \text{ m}$$

$$D = 0.0804 \text{ m} = 80.4 \text{ mm}$$

Case II:- $\tau_{\max} = 50 \text{ MPa}$

$$\frac{\tau_{\max}}{R} = \frac{\tau}{I_p}$$

$$\text{or, } \frac{50 \times 10^6}{R} = \frac{3580.986}{\frac{\pi}{2} R^4}$$

$$\text{or, } R = 0.0357 \text{ m}$$

$$D = 0.0714 \text{ m} = 71.45 \text{ mm}$$

Say, $D = 71.45 \text{ mm}$

Now,

$$\frac{T}{I_p} = \frac{G \phi}{L}$$

$$\text{or, } \frac{3580.986}{\frac{\pi}{2} \times (71.45 \times 10^{-3})^4} = \frac{1 \times 10^\circ \times \pi}{2}$$

$$\Rightarrow \phi = 1.6^\circ > 1^\circ \text{ } \{ \text{Not okay.} \}$$

So, $D = 80.4 \text{ mm}$ is req. dia.

6. A solid steel shaft has to transmit 100kW at 160 rpm. Taking allowable shear stress as 70MPa, find the suitable diameter of the shaft. The max. torque transmitted in each revolution exceeds the mean by 20%.

→ Solⁿ:

$$P = 100 \text{ kW}$$

$$f = 160 \text{ rpm}$$

$$\tau_{\text{max}} = 70 \text{ MPa}$$

$$d = ?$$

$$T_{\text{max}} = T_{\text{mean}} + 20\% \text{ of } T_{\text{mean}}$$

$$T_{\text{max}} = 1.2 T_{\text{mean}}$$

$$P = \frac{2\pi f T_{\text{mean}}}{60 \times 1000}$$

$$\text{or } \frac{100 \times 60 \times 1000}{2\pi \times 160} = T_{\text{mean}}$$

$$\Rightarrow T_{\text{mean}} = 5698.31 \text{ Nm}$$

$$T_{\text{max}} = 1.2 \times 5698.31 = 7111.97 \text{ Nm}$$

Now,

$$\frac{T_{\text{max}}}{I_p} = \frac{\tau_{\text{max}}}{R}$$

$$\text{or } \frac{7111.97}{\frac{\pi R^4}{2}} = \frac{70 \times 10^6}{R}$$

$$\Rightarrow R = 0.0402 \text{ m}$$

$$D = 80.47 \text{ mm}$$

7. A solid steel shaft has to transmit 100kW at 160 rpm. Taking allowable shear stress as 70MPa, find suitable diameter of shaft. The maximum torque transmitted in each revolution exceeds the mean by 20%.

→ Solⁿ:

$$P = 100 \text{ kW}$$

$$f = 160 \text{ rpm}$$

8. A hollow shaft having internal diameter 40% of its external diameter, transmits 562.5 kW power at 100 RPM. Determine the external diameter of the shaft if the shear stress is not exceed 60 N/mm² and twist in the length of 2.5 m should not exceed 1.3 degrees. Assume maximum torque to 1.25 times mean torque and modulus of rigidity as $9 \times 10^4 \text{ N/mm}^2$.

→ Solⁿ:

$$P = 562.5 \text{ kW}$$

$$N = 100 \text{ RPM}$$

$$\tau_s = 60 \text{ N/mm}^2$$

$$= 60 \times 10^6 \text{ N/m}^2$$

$$d = 0.4 D$$

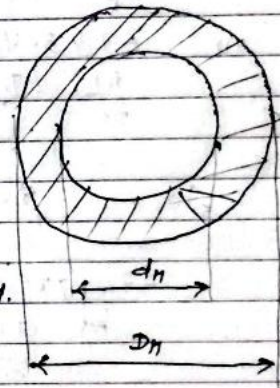
$$L = 2.5 \text{ m}$$

$$\phi = 1.3^\circ = 0.02269 \text{ rad.}$$

$$T_{\text{max}} = 1.25 T_{\text{mean}}$$

$$G = 9 \times 10^4 \times 10^6$$

$$= 9 \times 10^{10} \text{ N/m}^2.$$



$$P = \frac{2\pi f T_{\text{mean}}}{60 \times 1000} \text{ kW}$$

$$\Rightarrow T_{\text{mean}} = 53714.79 \text{ Nm.}$$

$T_{max} = 1.25 T_{mean} = 67143.49 \text{ Nm}$.
Case I: When shear stress limited to 60 N/mm^2
 We know,

$$\frac{T_{max}}{I_p} = \frac{\tau_s}{R}$$

$$\text{or, } T_{max} = \frac{2\tau_s}{D_H} \times \frac{\pi}{32} [D_H^4 - d_H^4]$$

$$\text{or, } 67143.9 = \frac{2 \times 60 \times 10^6 \times \pi}{32} \frac{[D_H^4 - (0.4D_H)^4]}{D_H}$$

$$D_H = 0.18017 \text{ m}$$

$$D_H = 180.17 \text{ mm}$$

Case II: Angle of twist not exceed 1.3°

$$\frac{T_{max}}{I_p} = \frac{G \phi}{L}$$

$$\text{or, } \frac{67143.49}{\frac{\pi}{32} (D_H^4 - d_H^4)} = \frac{9 \times 10^{10} \times 0.02269}{2.5}$$

$\downarrow$
 $0.4D_H$

$$\text{So, } D_H^4 = 171.21 \text{ mm}^4$$

So, Greater value of diameter should be adopted.

$$\therefore D_H = 180 \text{ mm}$$

□ Comparison of solid shaft & Hollow shaft:-

• comparison by strength:-

Let, shaft have same length, weight & same material

$$\text{Strength of Hollow shaft} = \frac{T_H}{T_S}$$

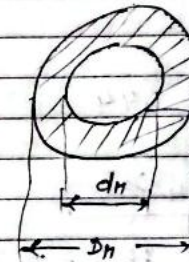
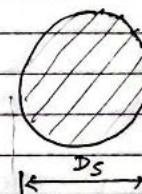
$$\text{Strength of Solid shaft} = \frac{T_S}{T_S}$$

$$\text{or, } \frac{T_H}{T_S} = \frac{\frac{\pi}{16} \tau_{max} (D_H^4 - d_H^4)}{\frac{\pi}{16} \tau_{max} D_S^4}$$

$$\text{or, } \frac{T_H}{T_S} = \frac{D_H^4 - d_H^4}{D_H D_S^3} \quad \text{--- (1)}$$

$$\text{Let, } \frac{D_H}{d_H} = n \Rightarrow D_H = n d_H$$

$$\frac{T_H}{T_S} = \frac{d_H^4 (n^4 - 1)}{n d_H D_S^3} = \frac{d_H^3 (n^4 - 1)}{n D_S^3} \quad \text{--- (2)}$$



Here,

✓ Weight of hollow = Weight of solid shaft

$$\text{or, } W_H = W_S$$

$$\text{or, } \gamma_H \cdot V_H = \gamma_S \cdot V_S \quad (\because \text{same material})$$

$$\text{or, } V_H = V_S$$

$$\text{or, } A_H L_H = A_S L_S \quad (\because \text{same length})$$

$$\text{or, } A_H = A_S$$

$$\text{or, } \frac{\pi}{4} (D_H^4 - d_H^4) = \frac{\pi}{4} D_S^4$$

$$\text{or, } n^4 d_H^4 - d_H^4 = D_S^4$$

$$\text{or, } D_S = d_H \sqrt[n^4 - 1]{n^4 - 1}$$

$$D_S^3 = d_H^3 (n^4 - 1)^{3/4} \quad \text{--- (3)}$$

Substituting (3) in (2), we get

$$\frac{T_H}{T_S} = \frac{n^4 - 1}{n (n^4 - 1)^{3/4}}$$

$$= \frac{(n^4 + 1)(n^4 - 1)}{n (n^4 - 1)^{3/4}}$$

$$\frac{T_H}{T_S} = \frac{n^4 + 1}{n \sqrt[n^4 - 1]{n^4 - 1}}$$

$$\therefore n = \frac{D_H}{d_H} > 1$$

$$\text{Let, } n = 2,$$

$$\frac{T_H}{T_S} = 1.44$$

$$\text{Let, } \Rightarrow \boxed{T_H > T_S}$$

Thus, strength of hollow shaft is greater than solid shaft for same weight, same material and same length.
{ Also, same x-section }

• Comparison by weight

Let shafts have same length, material and same torque. is applied.

Thus,

$$\frac{W_H}{W_S} = \frac{\gamma_H V_H}{\gamma_S V_S} = \frac{A_H L_H}{A_S L_S} = \frac{A_H}{A_S} \quad \left\{ \begin{array}{l} \text{same material} \\ \text{length} \end{array} \right.$$

$$\frac{W_H}{W_S} = \frac{D_H^2 - d_H^2}{D_S^2} \quad \text{--- (1)}$$

Let,

$$\frac{D_H}{d_H} = n \Rightarrow D_H = n d_H$$

$$\frac{W_H}{W_S} = \frac{d_H^2 (n^2 - 1)}{D_S^2} \quad \text{--- (2)}$$

Torque is same,

$$T_H = T_S$$

$$\text{or, } \frac{\pi}{16} \tau_{\max} \frac{(D_H^4 - d_H^4)}{D_H} = \frac{\pi}{16} \tau_{\max} D_S^3$$

$$\text{or, } \frac{d_H^4 (n^4 - 1)}{n d_H} = D_S^3$$

$$\text{or, } D_S^3 = \frac{d_H^3 (n^4 - 1)}{n}$$

$$\therefore D_S^2 = \frac{d_H^2 (n^4 - 1)^{2/3}}{n^{2/3}} \quad \text{--- (3)}$$

Substituting (3) in (2)

$$\frac{W_H}{W_S} = \frac{(n^4 - 1) n^{2/3}}{(n^4 - 1)^{2/3}} \quad \text{--- (4)}$$

$$\text{Let, } n = 2 \quad \{ n > 1 \}$$

$$\frac{W_H}{W_S} = 0.783$$

$$\Rightarrow \boxed{W_H < W_S} \quad \left\{ \begin{array}{l} \text{Hollow shaft} \\ \text{Economical} \end{array} \right.$$

Weight of hollow shaft will be smaller than weight of solid shaft for same length, material & same torque applied

□ Torsional Rigidity & Modulus of Rupture:

$$\text{Torsional Rigidity} = \frac{GJ}{L}$$

(Modulus of Rigidity)

We know, $T = \frac{G\phi}{J(L)}$

$$\text{or } T = \frac{GJ}{L} \phi$$

G, J, L are constants i.e. $T \propto \phi$
i.e.

Twisting moment $\propto$ Angle of twist.

• Torsional formula is only applicable upto elastic limits.

• If we determine strength by using this formula, then the strength will be fictitious.

• So, maximum fictitious shear stress calculated by the torsional formula using the experimentally found maximum torque required to rupture a shaft is called modulus of rupture in torsion.

⇒ Mathematically,

$$\text{Modulus of Rupture, } T_r = \frac{T_u \cdot R}{J}$$

here,

T_u = Ultimate torque at failure.

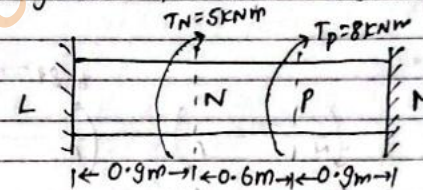
R = Outer radius of shaft

$$\text{fail } \frac{T_u}{J_p} = \frac{G\phi}{L} = \frac{T_r}{R} \quad \text{fictitious}$$

- since, the torsion equation is not applicable beyond limit of proportionality, T_r gives the fictitious value of shear stress at the ultimate torque.
- Although shear modulus of rupture is fictitious, it still can be used as a convenient tool for calculating maximum torque carrying capacity of shaft by using normal torsion equation.

Q8 Important

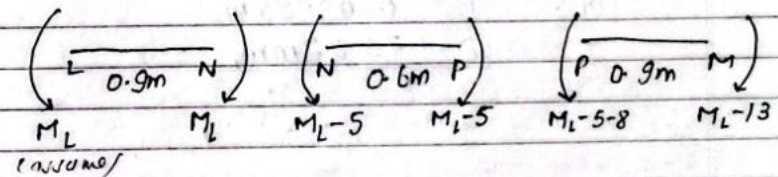
Figure shows a horizontal shaft LM subjected to axial twisting moments.



Determine

- The end fixing couples in magnitude & direction.
 - The diameter of shaft if the maximum shearing stress is not to exceed 80 MN/m^2 .
 - The position of section where shaft suffers no angular twist.
- Solⁿ: ∵ Both side fixed support, Angle of twist = 0
∵ Balancing torsions

(i)



$$\frac{T}{J_p} = \frac{q}{L}$$

Since $\phi = 0$

For series shafts

$$\phi_1 + \phi_2 + \phi_3 = \phi = 0$$

$$\text{or } \left(\frac{TL}{GJ_p}\right)_1 + \left(\frac{TL}{GJ_p}\right)_2 + \left(\frac{TL}{GJ_p}\right)_3 = 0$$

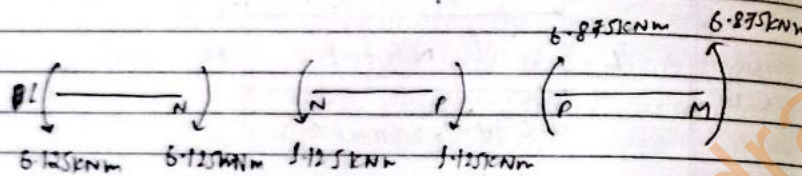
consider one side fixed (one dir) & see twist on other side; here all are clockwise if left end considered fixed for each section

$$\text{or } \frac{M_1 \times 0.9}{GJ_p} + \frac{(M_1 - 5) \times 0.6}{GJ_p} + \frac{(M_1 - 13) \times 0.9}{GJ_p} = 0$$

$$\text{or } 2.4 M_1 = 14.7$$

$$M_1 = 6.125 \text{ KNm}$$

Thus,



$$(ii) \tau_{max} = 80 \text{ MPa}$$

From Torsional Eqⁿ,

$$\frac{T}{J_p} = \frac{\tau_{max}}{R}$$

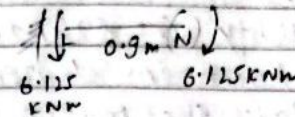
$$\text{or } \frac{6.875}{\frac{\pi}{2} R^4} = \frac{80 \times 10^3}{R}$$

$$\text{or } R = 0.03796 \text{ m}$$

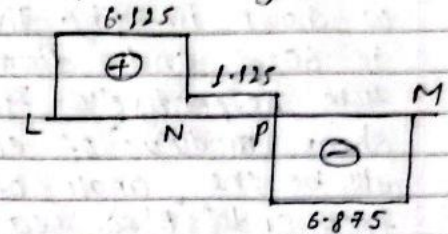
$$D = 75.92 \text{ mm}$$

(iii) Position of section for NO angular twist

Angle of twist at N



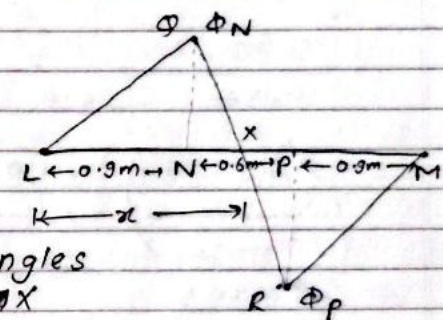
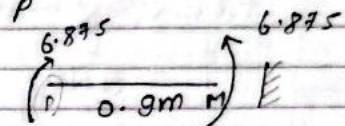
$$\phi_N = \frac{T_L L_1}{GJ} = \frac{6.125 \times 0.9}{GJ}$$



Torsion Distribution Diagram

ϕ Angle of twist at P

$$\phi_P = \frac{T_M L_3}{GJ} = \frac{-6.875 \times 0.9}{GJ}$$



From similar Triangles

$$\Delta \phi_N \approx \Delta P R X$$

$$\frac{\phi_N}{0.9} = \frac{\phi_P}{1.5 - x}$$

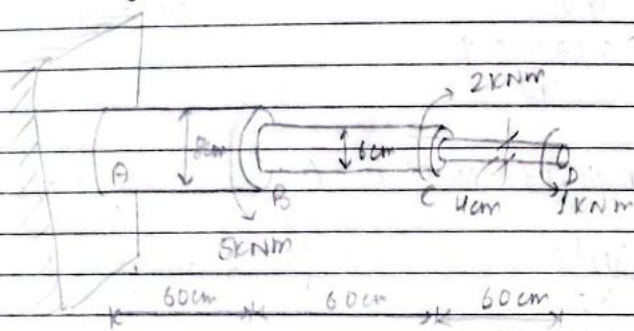
$$\text{or } \frac{6.125 \times 0.9}{x - 0.9} = \frac{-6.875 \times 0.9}{1.5 - x}$$

$$\text{or } 8.268 - 5.5125x = -6.1875x + 5.5687$$

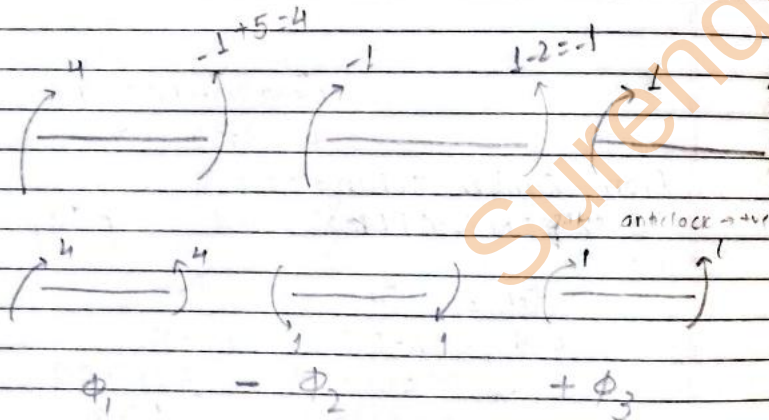
$$\text{on } 1.5 - x = -1.122(x - 0.9)$$

$$x = 1.183 \text{ m}$$

N9 A stepped shaft is subjected to torques as shown in Fig. The length of each section is 60cm and diameters are 8cm, 6cm and 4cm respectively. If the material has a shear modulus of elasticity $G = 80 \text{ GPa}$, what will be the angle of twist in degrees at free end? Also calculate the maximum shearing stress in shaft.



(See from one side say from D)



In series

$$\phi = \phi_1 - \phi_2 + \phi_3$$

$$= \frac{T_1 L_1}{G J_1} - \frac{T_2 L_2}{G J_2} + \frac{T_3 L_3}{G J_3}$$

$$= \frac{32 \times 10^{10}}{\pi \times 8 \times 10^{10} \times (10^{-2})^4} \left[\frac{4 \times 0.6}{8^4} - \frac{1 \times 0.6}{6^4} + \frac{1 \times 0.6}{4^4} \right]$$

$$= 0.031407 \text{ rad.}$$

Calculate τ for each segment & find $\tau_{\max}$ among them.

$$\tau_{AB} = \left(\frac{T \times R}{J_p} \right)_{AB} = \frac{4 \times 10^3 \times 4 \times 10^{-2}}{\frac{\pi}{32} \times (8 \times 10^{-2})^4}$$

$$= 39.79 \times 10^6 \text{ Pa}$$

$$\tau_{BC} = \left(\frac{T \times R}{J_p} \right)_{BC} = \frac{1 \times 10^3 \times 3 \times 10^{-2}}{\frac{\pi}{32} \times (6 \times 10^{-2})^4}$$

$$= 23.58 \times 10^6 \text{ Pa.}$$

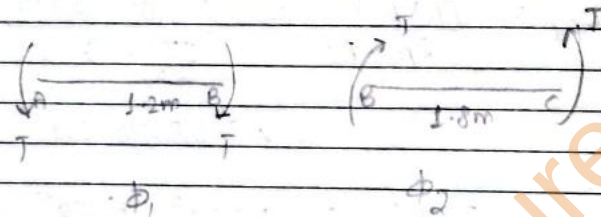
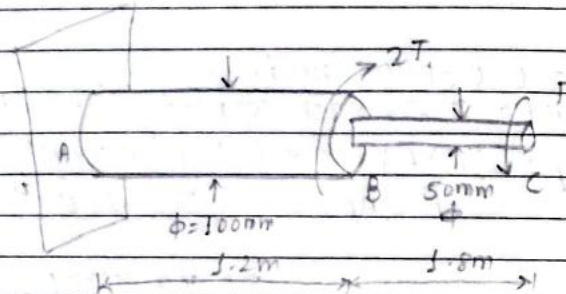
$$\tau_{CD} = \left(\frac{T \times R}{J_p} \right)_{CD} = \frac{1 \times 10^3 \times 2 \times 10^{-2}}{\frac{\pi}{32} \times (4 \times 10^{-2})^4}$$

$$= 79.58 \times 10^6 \text{ Pa.}$$

$$\text{Ther, } \tau_{\max} = \tau_{CD} = 79.58 \text{ MPa.}$$

N12 The stepped steel shaft subjected to a torque T at free end and a torque $2T$ is the opposite direction at the junction of two sizes. What is total angle of twist at free end, if the maximum stress in the shaft is limited to 70 MN/m^2 ? Assume modulus of rigidity to 84 GN/m^2 .

→ Solⁿ



$$\phi = \phi_1 - \phi_2$$

$$\phi = \left(\frac{TL}{GJ} \right)_{AB} - \left(\frac{TL}{GJ} \right)_{BC}$$

$$= \frac{2 \times 32 \times T}{(10^{-3})^4 \times 84 \times 10^9 \times \pi} \left[\frac{1.2}{100^4} - \frac{1.8}{50^4} \right]$$

$$= -3.3468 \times 10^{-5} \text{ rad.}$$

$$= 3.3468 \times 10^{-5} T \text{ (Anticlockwise dir.)}$$

$$\frac{T}{J} = \frac{\tau_{max}}{R} = \frac{G\phi}{L}$$

$$\text{For } \tau_{max} = 70 \text{ MN/m}^2$$

$$T = \frac{\pi}{16} \tau_{max} D^3$$

same for both

$$\tau_{max} \propto \frac{1}{D^3}$$

So,

τ_{max} exists for BC segment,

$$T = \frac{\pi}{16} \tau_{max} D^3$$

$$= \frac{\pi}{16} \times 70 \times 10^6 \times (0.05)^3$$

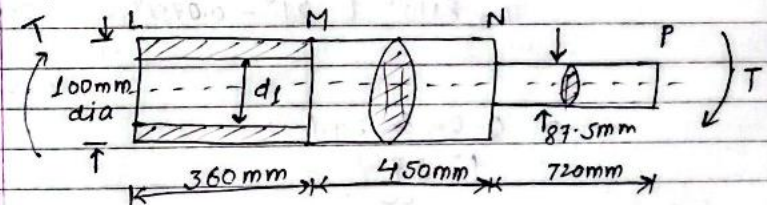
$$= 1718.06 \text{ Nm.}$$

Thus,

$$\phi = 0.0575 \text{ rad (5)}$$

N13

A steel shaft LMNP is made. If equal opposite torques are applied at the end of the shaft, find maximum permissible value of d_1 for maximum shearing stress in LM not to exceed that in NP. If torque applied is 10 kNm , what is total angle of twist? $G = 80 \text{ GN/m}^2$.



Sol: $T = 10 \text{ kNm}$

$G = 8 \times 10^{10} \text{ N/m}^2$

Condition:-

$(\tau_{\max})_{LM} = (\tau_{\max})_{NP}$

or, $\left(\frac{TR}{J_p}\right)_{LM} = \left(\frac{TR}{J_p}\right)_{NP}$

or, $\frac{50 \times 10^{-3} \times T}{\frac{\pi}{32} \left[\frac{100^4}{100} - \frac{d_1^4}{10^{-3}} \right] \times 10^{-3}} = \frac{T \times 87.5 \times 10^{-3}}{\frac{\pi}{32} \times (87.5 \times 10^{-3})^4}$

or,

or, $\frac{50 \times 10^{-3} \times T}{\frac{\pi}{32} \left[\frac{100^4}{100} - \frac{d_1^4}{10^{-3}} \right]} = \frac{87.5 \times 10^{-3} T}{\frac{\pi}{32} \times (87.5 \times 10^{-3})^4}$

or, $d_1 = 75.8 \text{ mm}$

For section,

Total angle of twist,

$\phi = \phi_1 + \phi_2 + \phi_3$

$= \left(\frac{TL}{GJ_p}\right)_1 + \left(\frac{TL}{GJ_p}\right)_2 + \left(\frac{TL}{GJ_p}\right)_3$

$= \frac{10 \times 10^3}{\pi \times 8 \times 10^{10}} \left[\frac{0.36}{0.1^4 - 0.075^4} + \frac{0.45}{0.1^4} + \frac{0.72}{0.0875^4} \right]$

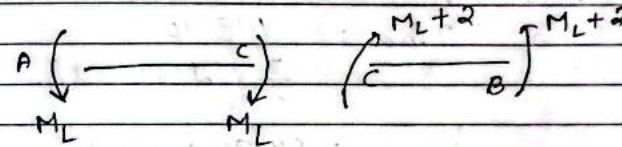
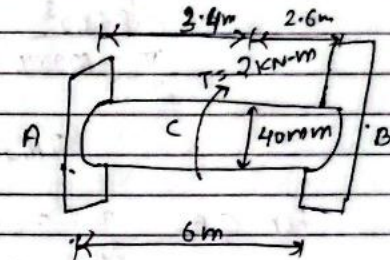
$= 0.0282 \text{ rad}$

$= 1^\circ 37'$

(10)

A solid steel shaft 6m long is fixed at each end. A torque of 2kN-m is applied to the shaft at a section 2.6m from right end. What are the fixing torques set up at the ends of the shaft? If the diameter of the shaft is 40 mm, what are the max. shearing stresses in the two portions? Calculate also the angle of twist for the section where the torque is applied. Take $G = 84 \text{ GPa}$. (Ans 90 MN/m², 68.96 MN/m², 8°)

Sol:



Fixed at both ends,

$\phi = 0$

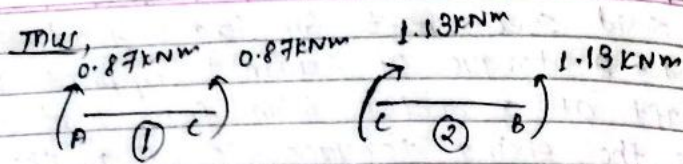
$\phi_1 + \phi_2 = 0$

or, $\left(\frac{TL}{GJ_p}\right)_1 + \left(\frac{TL}{GJ_p}\right)_2 = 0$

$J_{p1} = J_{p2}$

or, $M_L \times 3.4 + (M_L + 2) \times 2.6 = 0$

or, $M_L = \frac{-2 \times 2.6}{2.6 + 3.4} = -0.87 \text{ kNm}$



Max. shearing stresses,

$$\text{Max. } \tau_{AC} = ?$$

$$\text{Max. } \tau_{CB} = ?$$

From torsional equation,

$$\text{For AC, } \frac{T}{I_p} = \frac{\tau_{max}}{R}$$

$$\frac{0.87 \times 10^3}{\frac{\pi}{2} \times (20 \times 10^{-3})^4} = \frac{\tau_{max}}{20 \times 10^{-3}}$$

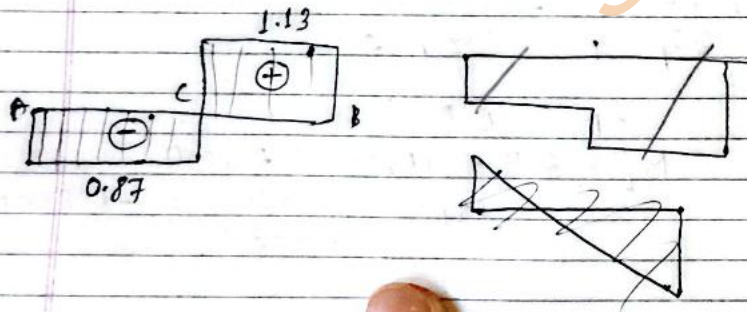
$$\Rightarrow (\tau_{max})_{AC} = 69.23 \text{ MN/m}^2$$

$$\text{For CB, } \frac{T}{I_p} = \frac{\tau_{max}}{R}$$

$$\text{or, } \tau_{max} = \frac{1.13 \times 10^3}{\frac{\pi}{2} \times (20 \times 10^{-3})^3}$$

$$= 89.92 \text{ MN/m}^2$$

Angle of twist at C = ?



$$\phi_c = \frac{T_{AC} L_{AC}}{G J_{AC}}$$

$$= \frac{0.87 \times 10^3 \times 23.4}{84 \times 10^9 \times \frac{\pi}{4} \times (20 \times 10^{-3})^4}$$

$$= 0.2802 \text{ rad}$$

$$= 16^\circ$$

(1) ~~Set~~ A solid cylindrical shaft is to transmit 300 kW at 100 rpm.

(i) If shear stress is not to exceed 80 MN/m², find diameter.

(ii) What percentage saving in wt. would be obtained if this shaft is replaced by a hollow one whose internal diameter equals 0.6 of external diameter; the length, material and maximum shear stress being the same?

Ans: D = 122 mm, 29.6%

→ Solution:-

$$P = 300 \text{ kW}$$

$$f = 100 \text{ rpm}$$

$$(i) \tau_{max} = 80 \text{ MPa}$$

$$P = \frac{2\pi f T}{60 \times 1000} \text{ kW}$$

$$\Rightarrow T = \frac{300 \times 60 \times 1000}{2\pi \times 100} = 28.648 \text{ kNm}$$

Now,

$$\frac{T}{I_p} = \frac{\tau_{max}}{R} \quad \left\{ \begin{array}{l} \text{Assume } \tau_{mean} = \tau_{max} \\ \text{or } \tau_{max} = 80 \times 10^6 \end{array} \right.$$

$$\text{on } \frac{28.648 \times 10^3}{\frac{\pi}{2} R^3} = 80 \times 10^6$$

$$\Rightarrow R = 61.08 \text{ mm}$$

$$\Rightarrow D = 122 \text{ mm}$$

(ii) $T_s = T_H$ { % saving in wt. }
 $d = 0.6 D$

$$T_H = \frac{\pi}{16} \times T_{max} \frac{(D^4 - d^4)}{D}$$

$$\text{or, } 28.648 \times 10^3 = \frac{\pi}{16} \times 80 \times 10^6 \frac{(D^4 - (0.6D)^4)}{D}$$

$$\Rightarrow D_H = 128 \text{ mm} \quad \left. \begin{array}{l} \text{Hollow} \\ \text{(for same } T_s \\ T_{max}) \end{array} \right\}$$

$$d = 76.78 \text{ mm}$$

$$\% \text{ saving in wt.} = \frac{Wt_{solid} - Wt_{hollow}}{Wt_{solid}} \times 100$$

$$= \frac{\frac{4}{3} \pi R_s^3}{\frac{4}{3} \pi R_s^3}$$

$$= \frac{(\frac{\pi D^2}{4}) \times L \times \rho - \frac{\pi (D_H^2 - d_H^2)}{4} \times L \times \rho}{\frac{\pi D^2}{4} \times L \times \rho}$$

$$= \frac{122^2 - (128^2 - 76.78^2)}{122^2} \times 100$$

$$= 29.53\%$$

Q. For a shaft of 60mm diameter, determine max. shear stress developed if twisting moment of 10kNm is applied on it. Also, calculate angle of twist in a 1.5m length of shaft if material made of steel for which $G = 90 \text{ GPa}$. Assume material is loaded up to elastic limit only.

→ Solⁿ:-

$$d = 60 \text{ mm}, T = 10 \text{ kNm}, L = 1.5 \text{ m}, G = 90 \text{ GPa}$$

$$\phi = ? \quad T_{max} = ?$$

We know,

$$\frac{T}{J} = \frac{T_{max}}{R} = \frac{\tau}{\rho} = \frac{G \phi}{L}$$

For circular shaft,

$$J = \frac{\pi D^4}{32} = 1.272 \times 10^{-6} \text{ m}^4$$

$$T_{max} = \frac{T}{J} \times R$$

$$= \frac{10 \times 10^3 \times 30 \times 10^{-3}}{1.272 \times 10^{-6}}$$

$$= 235.85 \text{ MN/m}^2$$

$$\phi = \frac{T L}{G J} = \frac{10 \times 10^3 \times 1.5}{90 \times 10^9 \times 1.272 \times 10^{-6}} = 0.131 \text{ rad}$$

$$= 7.51^\circ$$

Q For a shaft angle of twist in a length of 25m diameter must not exceed 1.2° and shear stress should be less than 60 N/mm^2 . If the shaft has to transmit a tangential force of 1.3 MN , determine actual working stress and diameter of shaft. Take $G = 9 \times 10^4 \text{ N/mm}^2$.

→ Soln:

$$L = 25 \text{ D}$$

$$G = 90 \text{ GN/m}^2$$

$$\phi_{\max} = 1.2^\circ = 0.02094 \text{ rad.}$$

$$(\tau_{\max})_{\text{permissible}} = 60 \text{ N/mm}^2, F = 1.3 \text{ MN}$$

$$\text{Torque, } T = F \times R = 1.3 \times 10^6 \times \frac{D}{2}$$

$$T = 0.65 \text{ D MNm}$$



Case I: $\phi_{\max} = 1.2^\circ = 0.02094 \text{ rad.}$

$$\frac{T}{J} = \frac{G \phi}{L}$$

$$\text{or, } \frac{0.65 \times 10^6 \text{ D}}{\frac{\pi}{32} \times D^4} = \frac{90 \times 10^9 \times 0.02094}{25 \text{ D}}$$

$$\text{or, } D = 0.296 \text{ m} = 296.36 \text{ mm.}$$

Now,

$$\frac{\tau_{\max}}{R} = \frac{T}{J} \quad \left\{ (\tau_{\max})_{\text{actual}} = \frac{16T}{\pi D^3} \right\}$$

$$\tau_{\max} = \frac{0.65 \times 10^6 \times 0.296}{\frac{\pi}{32} \times (0.296)^4} \times \frac{0.296}{2}$$

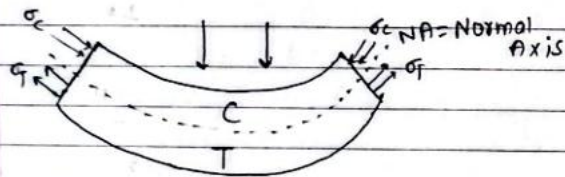
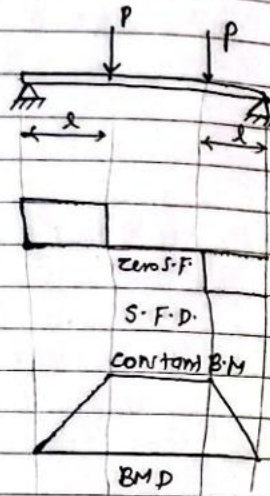
$$\text{Max. Shear stress, } (\tau_{\max})_{\text{actual}} = 37783313.74 \text{ N/m}^2 = 37.78 \text{ MN/m}^2$$

$$(\tau_{\max})_{\text{permissible}} > (\tau_{\max})_{\text{actual}} \quad \text{Hence, OK (can resist).}$$

- Coplanar and Pure Bending
- If the bending of beam takes place due to constant bending moment and it is free from shear force then this condition is called pure bending or flexure.

→ The normal stress due to bending and flexure stress.

→ In figure, the mid portion is free from shear force and subjected to constant BM.

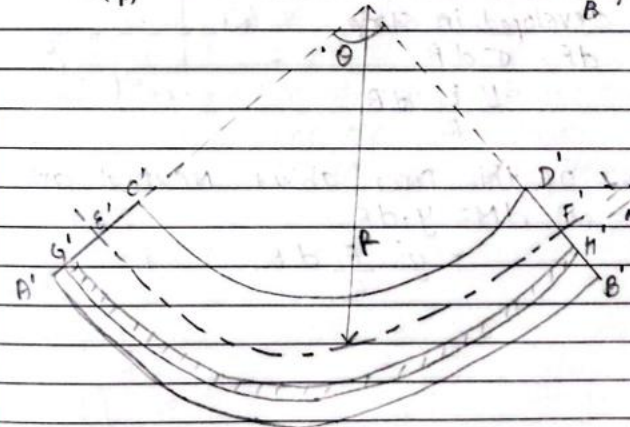
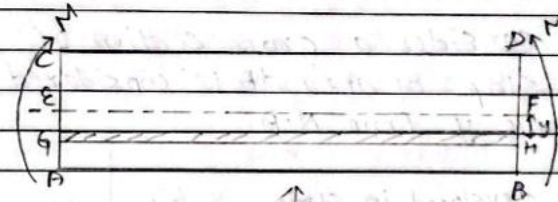


→ In NA, zero stress, i.e. NA line neither in tension nor compression

Derivation of Flexure Equation:

Assumptions

- (1) The material of beam is perfectly homogeneous throughout.
- (2) The stress induced is proportional to the strain and stress doesn't exceed elastic limit.
- (3) The value of modulus of Elasticity E is same for whole beam either in compression or in tension.
- (4) The transverse section of the beam which is plane before bending remains plane after bending. (No warping)
- (5) There is no resultant push or pull on the cross section of beam.
- (6) The loads are applied in plane of bending.



Consider a beam subjected to bending moment M , as shown in figure due to bending the beam will be bent as shown.

Let us consider strip GH at a distance y from neutral axis. The length of strip GH will be changed after bending. So, strain produced in GH

$$E_{GH} = \frac{GH' - GH}{GH} = \frac{(R+y)\theta - R\theta}{R\theta}$$

$$E_{GH} = \frac{y}{R}$$

Let, σ be the stress developed at strip GH, then

$$\sigma = E_{GH} E = \frac{E y}{R} \quad \text{[Hooke's law]}$$

$$\left[\frac{\sigma}{y} = \frac{E}{R} \right] \quad \text{--- (1)}$$

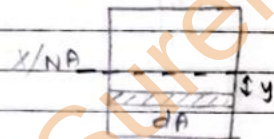
y : from Neutral axis either upward or downward

For given material & given bending, E & R const.

Now, consider a cross section of beam. Let the strip of area dA is considered at distance y from N.A.

(Resulting force)

$$\begin{aligned} \text{Force developed in strip, } dF &= \sigma dA \\ &= \frac{y E}{R} dA \end{aligned}$$



Moment of this force about Neutral axis dM

$$\begin{aligned} \text{(Resulting Moment) } dM &= y \cdot dF \\ &= y^2 \frac{E}{R} dA \end{aligned}$$

$$\left. \begin{aligned} \sum F &= 0 \quad (\text{Beam stationary}) \\ F &= \int y \frac{E}{R} dA = 0 \\ \frac{E}{R} \int y dA &= 0 \\ \Rightarrow \int y dA &= 0 \\ \Rightarrow \text{Neutral axis} &= \text{Centroidal axis} \end{aligned} \right\}$$

Total Moment developed

$$M = \int y^2 \frac{E}{R} dA \quad \left\{ \begin{array}{l} \text{In eqn, } M_R = M \end{array} \right.$$

$$\text{or, } M = \frac{E}{R} \int y^2 dA$$

$$\text{or, } M = \frac{E I}{R} \quad \left\{ \begin{array}{l} M I \text{ about Neutral axis.} \\ I_x / I_{NA} \end{array} \right.$$

$$\text{or, } \frac{M}{I} = \frac{E}{R} \quad \text{--- (2)} \quad \left\{ \begin{array}{l} \text{Centroidal axis} \end{array} \right.$$

$$\therefore \left[\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \right] \text{ is flexure equation}$$

Flexural stiffness

$$\frac{M}{I} = \frac{E}{R}$$

For particular beam, $I \rightarrow \text{const}$

For given material, $E \rightarrow \text{const}$

$$M R = \frac{E I}{\text{const.}}$$

$E I \rightarrow$ Flexural stiffness

$$M \propto \frac{1}{R}$$

Moment inversely proportional to radius of curvature.
(within elastic limit)

Q.1 A 250mm x 150mm rectangular beam is subjected to max BM of 750 kNm. Determine max stress in the beam. If value of E is 200 GN/m². Find out radius of curvature for the position of beam when bending is max. Find value of longitudinal stress at a distance 65mm from top surface of beam.

→ Soln:

$$M = 750 \text{ kNm}$$

$$\sigma = ?$$

$$R = ?$$

$$\sigma_{65\text{mm from top}} = ?$$

Flexure equation,

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\text{For, max } M, y = 125 \text{ mm}$$

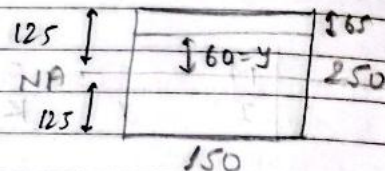
$$\sigma = \frac{750 \times 10^3 \times 125 \times 10^{-3}}{\frac{(150 \times 10^{-3}) \times (250 \times 10^{-3})^3}{12}}$$

$$= 480 \text{ MPa.}$$

$$R = \frac{E y}{\sigma}$$

$$= \frac{2 \times 10^{11} \times 125 \times 10^{-3}}{480 \times 10^6}$$

$$= 52.08 \text{ m.}$$



(In beam, depth = σ)

$$\frac{bd^3}{12} \quad d \uparrow \quad I \uparrow$$

More resists bending to moment

$$\text{For, } y = 125 - 65 = 60 \text{ mm}$$

$$\sigma = \frac{E y}{R}$$

$$= \frac{2 \times 10^{11} \times 60 \times 10^{-3}}{52.08}$$

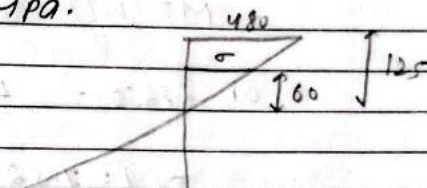
$$= 230.42 \text{ MPa.}$$

or

From sim Δ ,

$$\frac{\sigma}{480} = \frac{60}{125}$$

$$\Rightarrow \sigma = 230.4 \text{ MPa.}$$



Q.2 A symmetrical section 200mm deep has a MOI of $2.26 \times 10^{-5} \text{ m}^4$ about neutral axis. Determine longest span over which, when simply supported beam would carry a uniformly distributed load of 4 kN/m run without the bending stress exceeding 125 MPa.

→ Soln:

$$I = 2.26 \times 10^{-5} \text{ m}^4$$

$$d = 200 \text{ mm}$$

$$y = 100 \text{ mm}$$

$$\sigma_{\text{max}} = 125 \text{ MPa.}$$



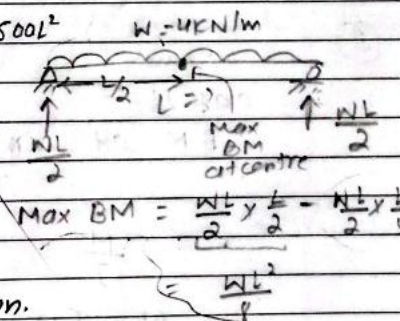
$$M = \frac{WL^2}{8} = \frac{4 \times 10^3 \times L^2}{8} = 500L^2$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\text{or, } \frac{500L^2}{2.26 \times 10^{-5}} = \frac{125 \times 10^6}{100 \times 10^{-3}}$$

$$\text{or, } L = 56.5 \text{ m}$$

$$L = 56.5 \text{ m} \quad \text{Longest span.}$$



$$\frac{M}{I} = \frac{\sigma_{\max}}{y}$$

$$\text{or } \frac{M}{2.26 \times 10^{-5}} = \frac{125 \times 10^6}{0.1}$$

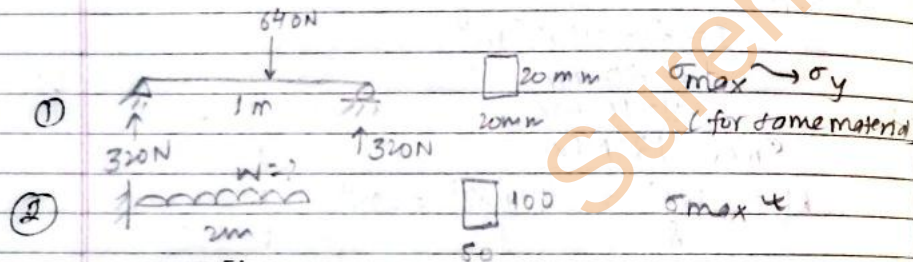
$$\Rightarrow M = 28250 \text{ Nm} = 28.25 \text{ kNm}$$

$$M = \frac{WL^2}{8}$$

$$\text{or } 28250 = \frac{4 \times 10^3 \times L^2}{8}$$

$$\Rightarrow L = 7.52 \text{ m}$$

- (4) A cast iron test beam of cross section 20mm * 20mm and 1m long supported at the ends fail when central load of 640N is applied. What UDL will break a cantilever beam of same material and 50mm * 100mm cross section and 2m long?



Case I:

$$\text{BM at centre, } M_{\max} = 320 \times 0.5 = 160 \text{ Nm}$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\text{or } \sigma_{\max} = \frac{160 \times 10 \times 10^{-3}}{20 \times 10^{-3} \times (20 \times 10^{-3})^3} = 120 \text{ MPa}$$

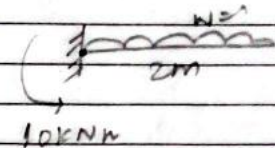
Case II: Cantilever Beam,

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\text{or } M = \frac{\sigma}{y} I = \frac{120 \times 10^6 \times 50 \times 10^{-3}}{50 \times 10^{-3} \times (100 \times 10^{-3})^3} \times \frac{1}{12}$$

$$M_{\max} = 10 \text{ kNm}$$

For cantilever beam, $M_{\max}$ at fixed support,



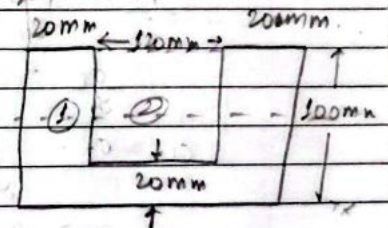
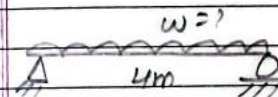
$$M = W \times 2 \times 1$$

$$\text{or } 10 \times 10^3 = W \times 2 \times 1$$

$$\text{or } W = 5 \text{ kN/m}$$

5. The horizontal beam is 4m long and is simply supported at ends. Calculate max UDL it can carry if the tensile and compressive stresses must not exceed 25MPa and 45MPa respectively.

Solⁿ:



$$\left. \begin{aligned} (\sigma_T)_{\max} &= 25 \text{ MPa} \\ (\sigma_C)_{\max} &= 45 \text{ MPa} \end{aligned} \right\} \text{Might not occur simultaneously}$$

$$\begin{aligned}\bar{y} &= 35 \text{ mm} \\ y_t &= 35 \text{ mm} \\ y_c &= 100 - 35 = 65 \text{ mm} \\ I_{xx} &= 581.33 \times 10^{-8} \text{ m}^4 \\ &= 5813332 \text{ mm}^4\end{aligned}$$

$$\bar{x} = 80 \text{ mm}$$

$$\begin{aligned}\bar{y} &= \frac{50 \times 100 \times 160 - 60 \times 120 \times 80}{100 \times 160 - 120 \times 80} \\ &= 35 \text{ mm}\end{aligned}$$

$$\begin{aligned}I_{xx} &= (I_{xx})_1 - (I_{xx})_2 \\ &= 160 \times 100\end{aligned}$$

For simply supported beam,

$$M_{\max} = \frac{WL^2}{8} = \frac{W \times 16}{8}$$

$$M = 2W$$

$$\frac{M}{I} = \frac{\sigma_t}{y_t} = \frac{\sigma_c}{y_c}$$

$$\frac{\sigma_t}{y_t} = \frac{\sigma_c}{y_c}$$

$$\sigma_t = \frac{y_t}{y_c} \sigma_c$$

$$\sigma_t = 0.538 \sigma_c$$

$$\begin{aligned}\text{Take } \sigma_t &= 25 \\ \sigma_c &= 46 > 45 \\ &\text{Not ok}\end{aligned}$$

$$\begin{aligned}\text{Take } \sigma_c &= 45 \\ \sigma_t &= 24.7 < 25 \\ &\text{(Ok)}\end{aligned}$$

Thus,

$$\begin{aligned}\text{Take } \sigma_c &= 45 \text{ MPa} \\ \sigma_t &= 24.7 \text{ MPa}\end{aligned} \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} \text{Actual } \sigma_c \\ \text{Actual } \sigma_t \end{array} \begin{array}{l} \text{max} \\ \text{developed.} \end{array}$$

Now,

$$\frac{M}{I} = \left(\frac{\sigma_c}{y_c} \right)_{\text{Actual}} = \left(\frac{\sigma_t}{y_t} \right)_{\text{Actual}}$$

$$\text{Or, } \frac{2W}{I} = \frac{45}{65}$$

$$\text{Or, } W = \frac{45 \times 10^6 \times 581.33 \times 10^{-8}}{65 \times 10^{-3} \times 2}$$

$$W = 2.012 \text{ kNm}$$

□ Built-Up Beam

→ Section of beam made up of two or more materials joined together in such a way that they behave like a single piece.

→ Same radius of curvature

$$R_1 = R_2 \quad \rightarrow \text{For No deformation.}$$

→ Total moment is resisted by two materials.

$$M = M_1 + M_2$$

→ For material 1, $\rightarrow \frac{M_1}{I_1} = \frac{\sigma_1}{y_1} = \frac{E_1}{R_1}$

For material 2, $\rightarrow \frac{M_2}{I_2} = \frac{\sigma_2}{y_2} = \frac{E_2}{R_2}$

→ Since, $\frac{1}{R_1} = \frac{1}{R_2}$

$$\Rightarrow \frac{M_1}{E_1 I_1} = \frac{M_2}{E_2 I_2}$$

$$\Rightarrow \frac{\sigma_1}{E_1 y_1} = \frac{\sigma_2}{E_2 y_2}$$

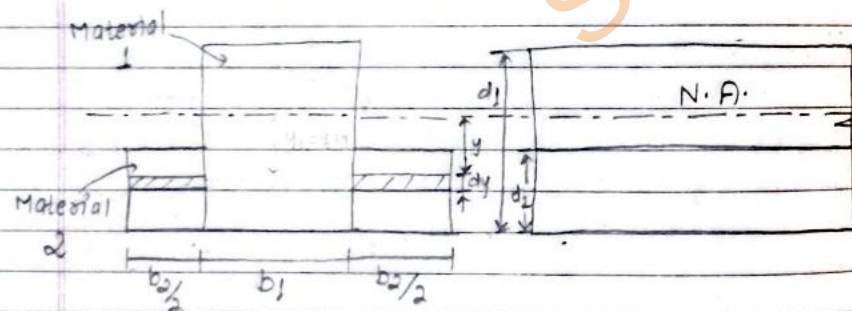
$$\Rightarrow \frac{\sigma_2}{\sigma_1} = \frac{E_2}{E_1} \cdot \frac{y_2}{y_1}$$

If $y_1 = y_2$ then,

$$\frac{\sigma_2}{\sigma_1} = \frac{E_2}{E_1} = m$$

$m \rightarrow$ Modular Ratio

N.A. $\rightarrow$ Neutral Axis.



Consider resultant force dF_2 in an element of material 2 having thickness dy & breadth b_2 located at distance y from N.A.

$$dF_2 = \sigma_2 (b_2 dy)$$

$$= m \sigma_1 (b_2 dy)$$

$$= (m b_2) (\sigma_1 dy)$$

$\therefore$ Material 2 can be transformed into material 1 by multiplying the original breadth by m .

(depth no change as position of N.A. changes)

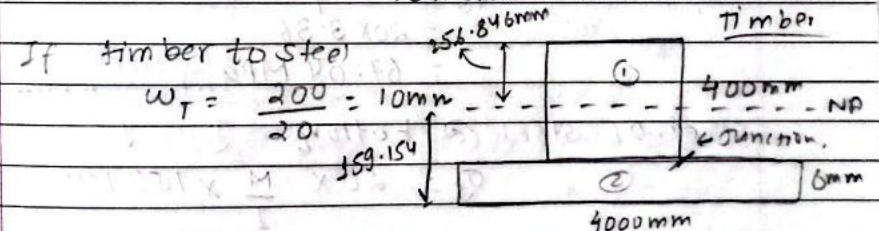
Find the extreme fiber stress given that maximum bending moment is 40 kNm and modular ratio is 20.

→ Solⁿ:

$m = 20 = \frac{\text{Steel}}{\text{Timber}}$	timber	400mm
$\frac{\sigma_s}{\sigma_T} = \frac{E_s}{E_T} = 20 \quad (E_s > E_T)$	Steel	6mm
$\sigma_s = m \cdot \sigma_T$	100mm	

Transforming Steel to timber.

New width of Steel timber = $m \times$ Initial width of Steel
 $= 20 \times 200 = 4000 \text{ mm}$



calculating $\bar{y}$ & I_{xx} (bottom as ref.)

Component	$\bar{y}_i$	A_i	$y_i A_i$
①	206	80000	16.48×10^6
②	3	24000	72×10^3

$\Sigma A_i = 104 \times 10^3$ $\Sigma y_i A_i = 16.552 \times 10^6$

$$\bar{y} = \frac{\Sigma y_i A_i}{\Sigma A_i} = 159.154 \text{ mm.}$$

Stress of timber at the top

$$\frac{M}{I} = \frac{\sigma}{y} \Rightarrow \sigma_T = \frac{M}{I} \times 246.846$$

transformed section (of timber)

$$= \frac{40 \times 10^3 \times 246.846 \times 10^{-3}}{1.8275 \times 10^9 \times 10^{-12}}$$

so, σ of timber = 5.403 MPa

Stress of timber at the bottom junction,

$$\Rightarrow \sigma = \frac{M}{I} \times (159.154 - 6) \times 10^{-3}$$

(of timber)

$$= \frac{40 \times 10^3 \times 153.154 \times 10^{-3}}{1.8275 \times 10^9 \times 10^{-12}}$$

$$= 3.35 \text{ MPa}$$

Stress of steel @ junction,

for steel $\sigma_s = m \sigma_t$

$$= 20 \times 3.36$$

$$= 67.04 \text{ MPa. of steel}$$

Stress of steel @ bottom

$$\sigma_s = 20 \times \frac{M}{I} \times 159.154$$

$$= 69.87 \text{ MPa}$$

How much UDL can this beam carry if it is used as a simply supported beam of 6m span?

sol:

$$m = 20 = \frac{\sigma_s}{\sigma_t} = \frac{E_s}{E_t}$$

Transforming steel section to timber,

New width = $m \times$ initial width

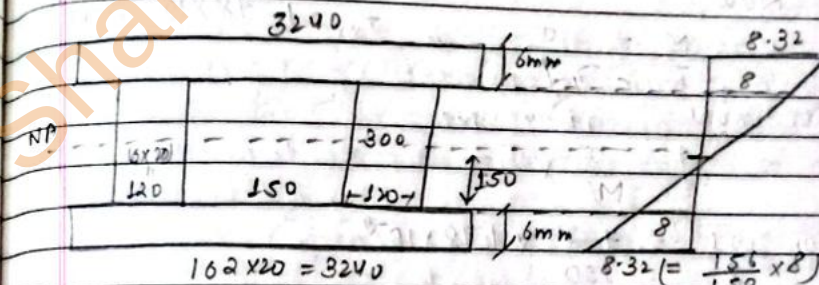


Fig: Transformed section

Fig: Stress diagram.

Taking $\sigma_t = 8 \text{ MPa}$

$$\sigma_{\text{top/bottom}} = \frac{156}{150} \times 8$$

$$= 8.32 \text{ MPa}$$

At extreme fiber
For timber $\rightarrow$ extreme fiber at 150 mm from NA.

$\sigma_t, \text{max} = 8 \text{ MPa.}$

Corresponding

stress for steel; $\sigma_s = 8.32 \times 20$

(at edge max) = 166 MPa $\neq 150 \text{ MPa}$

Not Okay.

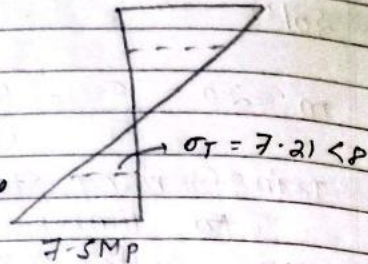
Taking $\sigma_s = 150 \text{ MPa}$ } At top/bottom
corresponding value for timber,
 $\sigma_t = \frac{150}{20} = 7.5 \text{ MPa}$

Max value for timber
(at junction)

$$\sigma_t = \frac{150}{156} \times 7.5$$

$$= 7.21 \text{ MPa} < 8 \text{ MPa}$$

(okay)



Now,

$$\frac{M}{I} = \frac{\sigma}{y}$$

For timber

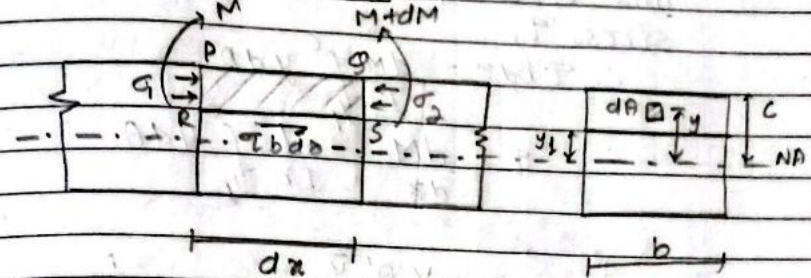
$$M = \frac{\sigma_t \times I}{y}$$

$$\text{or, } \frac{w l^2}{8} = \frac{7.21 \times 1.78 \times 10^{-9}}{150}$$

$$\text{or, } \frac{w \times (6 \times 10^{-3})^2}{8} = \frac{7.21 \times 10^3}{150 \times 10^{-3}} \times 1.78 \times 10^{-9} \times 10^{-12}$$

$$w = 19.01 \text{ MN/m}^2$$

Horizontal Shear Stress



(lengthwise)

Take a layer at a distance y_1 from N.A. Let the width of this layer be ' b ', and its area be ' $b dy$ '. Stress at the face PR will be slightly less than face QS. This excessive force will be balanced by the shear force acting along the layer RS. Then, total force on face PR is given by

$$F_{PR} = \int_{y_1}^c \sigma_s dA = \int_{y_1}^c \frac{M y}{I} dA \quad \left\{ \begin{array}{l} \text{From flexure} \\ \text{eq.} \end{array} \right.$$

And on face QS is given by

$$F_{QS} = \int_{y_1}^c \frac{(M+dM) y}{I} dA \quad \left\{ \begin{array}{l} \sigma_2 > \sigma_1 \\ \sigma_{QS} > \sigma_{PR} \end{array} \right.$$

The excess force between PR & QS is

$$dF = \int_{y_1}^c \frac{dM}{I} y dA \quad \left\{ \begin{array}{l} \text{excess moment} \\ dM \end{array} \right.$$

$$F_{QS} - F_{PR} = \frac{dM}{I} \int_{y_1}^c y dA$$

This excess force is balanced by shear stress τ ,

$$\therefore \tau b dx = \frac{dM}{I} \int_{y_1}^c y dA$$

$$\tau = \frac{dM}{dx} \cdot \frac{1}{Ib} \int_{y_1}^c y dA$$

$$= \frac{V A' \bar{y}}{Ib}$$

$$\tau = \frac{VS}{Ib}$$

MS of
cls

V is vertical shear force.

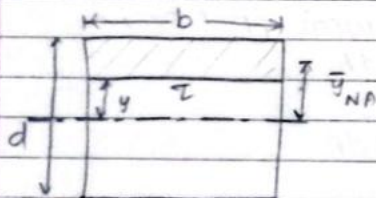
S = Sum of moments of differential area dA about NA

A' = partial area of sections above layer at which τ is computed

$\bar{y}$ is distance from centroid of considered layer to Neutral axis.

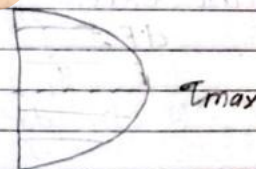
□ Variation of shear stress in a beam cross section.

① Rectangular section:-



cls

Shear stress distribution



$$\tau = \frac{V A' \bar{y}}{Ib}$$

where, $A' = b \left(\frac{d}{2} - y \right)$

$$\bar{y} = y + \frac{1}{2} \left(\frac{d}{2} - y \right)$$

So,

$$\tau = \frac{V}{Ib} \left[b \left(\frac{d}{2} - y \right) \right] \left[y + \frac{1}{2} \left(\frac{d}{2} - y \right) \right]$$

$$= \frac{V}{Ib} \left[\frac{d}{2} y - y^2 + \frac{1}{2} \frac{d^2}{4} - \frac{dy}{4} - \frac{dy}{4} + \frac{y^2}{2} \right]$$

$$= \frac{V}{2I} \left[\frac{d^2}{4} - y^2 \right]$$

→ Parabolic along depth of section

When $y = 0$; $\tau_{max} = \frac{V}{2I} \times \frac{d^2}{4}$

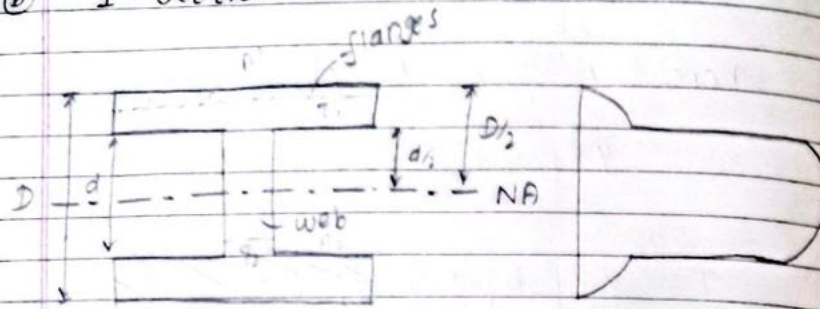
$$= \frac{V}{2 \left(\frac{bd^3}{12} \right)} \times \frac{d^2}{4}$$

$$= 1.5 \frac{V}{b d} \rightarrow \text{vertical shear force} / \text{area}$$

Thus, $\tau_{max} = 1.5 \tau_{avg}$

I-section
flange $\rightarrow$ B.M (resist)
web $\rightarrow$ resist S.F
Date _____
Page _____

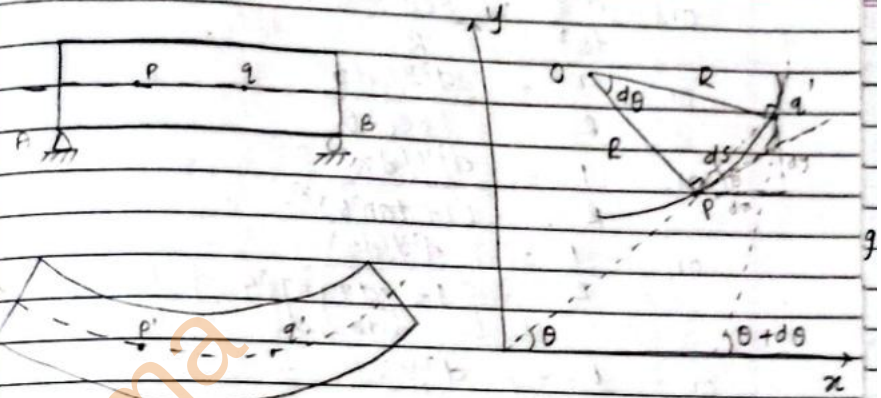
(b) I-section



Shear Stress distribution
diagram

Date _____
Page _____

Deflection of Beam:



Draw tangents at P' and Q' .

Let θ be angle made by tangent at P' &

$\theta + d\theta$ be angle made by tangent at Q'

R is Radius of curvature.

From figure,

$$ds = R d\theta$$

$$\Rightarrow \frac{1}{R} = \frac{d\theta}{ds} \quad \text{--- (1)}$$

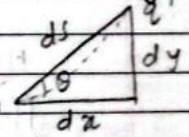
As $ds \rightarrow 0$, $P'Q'$ is approximately a straight line having components dx & dy .

$$\tan \theta = \frac{dy}{dx}$$

Differentiating w.r.t. x ,

$$\sec^2 \theta \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\begin{aligned} \text{On } \frac{d^2y}{dx^2} &= \sec^2 \theta \frac{d\theta}{ds} \frac{ds}{dx} \\ &= \sec^2 \theta \frac{1}{R} \sec \theta \end{aligned}$$



Flexure eqⁿ: $\left[\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \right] \Delta$

or, $\frac{d^2y}{dx^2} = \frac{\sec^3 \theta}{R}$

or, $\frac{1}{R} = \frac{d^2y/dx^2}{\sec^3 \theta}$

or, $\frac{1}{R} = \frac{d^2y/dx^2}{(1 + \tan^2 \theta)^{3/2}}$

or, $\frac{1}{R} = \frac{d^2y/dx^2}{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}} \quad \left[\frac{dy}{dx} \rightarrow \text{small quantity} \right]$

or, $\frac{1}{R} = \frac{d^2y}{dx^2}$

or, $\frac{M}{EI} = \frac{d^2y}{dx^2}$

$M = EI \frac{d^2y}{dx^2} \quad \text{--- (2)}$

This is called differential equation for deflection.

Integrating eqⁿ (2) {Diff^l eqⁿ}

$\int M = EI \frac{dy}{dx} + C_1$
↓
Slope

Again,

$\int \int M = EI y + C_1 x + C_2$
↓
deflection

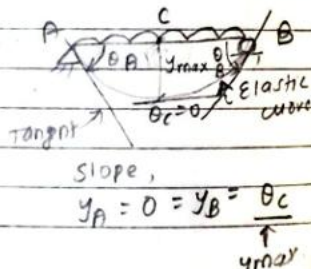
For finding constants (C_1, C_2) :

Boundary conditions

For simply supported beam,

$y_A = 0 = y_B = \theta_C$

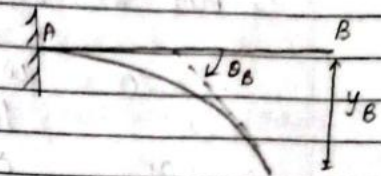
↑
 y_{max}



• For cantilever beam

$y_A = 0$

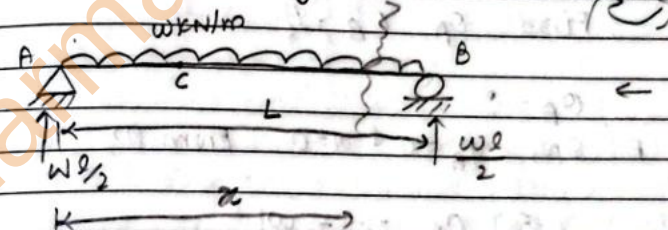
$\theta_A = 0$



□ Methods of Finding Deflections:

① Double integration method

↳ Single Loads.



Moment at x ,

$M_x = \frac{wL}{2} x - \frac{wx^2}{2}$

or, $EI \frac{d^2y}{dx^2} = \frac{wL}{2} x - \frac{wx^2}{2}$

Integrating

or, $\left[EI \frac{dy}{dx} = \frac{wL}{2} \frac{x^2}{2} - \frac{wx^3}{6} + C_1 \right] \text{--- (I)}$

Again on integrating,

$\left[EI y = \frac{wL}{12} \frac{x^3}{3} - \frac{wx^4}{24} + C_1 x + C_2 \right] \text{--- (II)}$

Boundary condition

$y_A = 0 \rightarrow \text{at } x = 0$

$y_B = 0 \rightarrow \text{at } x = L$

Put $x = 0, y = 0$ @ eqⁿ (I)

$[C_2 = 0]$

Put $x = L/2$, $\theta = 0 = \frac{dy}{dx}$ in eqⁿ ①

$$0 = \frac{wL^3}{16} - \frac{wL^3}{48} + C_1$$

$$\text{or, } C_1 = \frac{wL^3}{48}$$

$$[C_1 = -\frac{wL^3}{24}]$$

Thus,

Find θ_A , θ_B & y_c

$\theta_A = ?$

At θ_A , $y_A = 0$ & $x = 0$ From ①,

$$EI \theta_A = -\frac{wL^3}{24}$$

$$[\theta_A = \frac{-wL^3}{24EI}]$$

For θ_B , $y_B = 0$ & $x = L$

From ①

$$EI \theta_B = \frac{wL^3}{4} - \frac{wL^3}{6} - \frac{wL^3}{24}$$

$$\theta_B = \frac{wL^3}{24EI}$$

For y_c , $x = L/2$

From ①

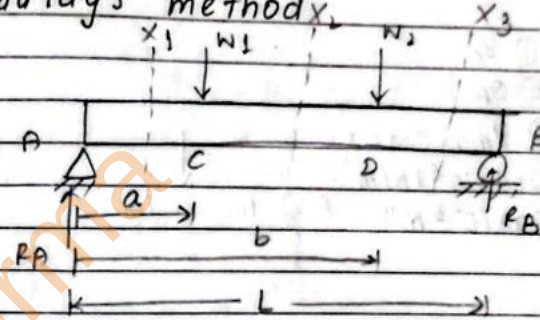
$$EI y_c = \frac{wL^4}{12 \times 8} - \frac{wL^4}{24 \times 16} - \frac{wL^4}{24 \times 2}$$

$$y_c = \frac{-5L^4}{384EI}$$

$$[y_c = \frac{wL^3}{48EI}]$$

y : -ve, deflection down
 y : +ve, deflection up.

② Macaulay's method, For number of loads.



At section x_1 , (between A & C)

$$M_x = R_A \cdot x \quad \{0 < x < a\}$$

At section x_2 , (between C & D)

$$M_x = R_A \cdot x - W_1(x-a) \quad \{0 < x < b\}$$

At section x_3 , (between D & B)

$$M_x = R_A \cdot x - W_1(x-a) - W_2(x-b) \quad \{0 < x < L\}$$

In general for any section B.M. is:

$$M_x = EI \frac{d^2y}{dx^2} = R_A x - W_1 \langle x-a \rangle - W_2 \langle x-b \rangle$$

Integrating,

$$EI \frac{dy}{dx} = R_A \cdot \frac{x^2}{2} + C_1 - W_1 \frac{\langle x-a \rangle^2}{2} - W_2 \frac{\langle x-b \rangle^2}{2}$$

$$[EI y = \frac{R_A x^3}{6} + C_1 x + C_2 - W_1 \frac{\langle x-a \rangle^3}{6} - W_2 \frac{\langle x-b \rangle^3}{6}]$$

* Generally, write C_1 & C_2 after first term

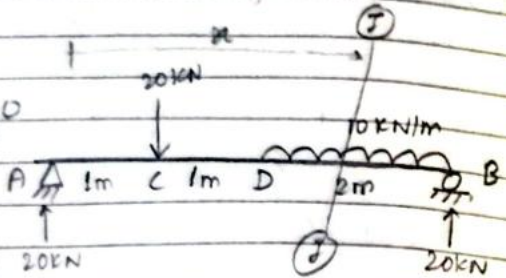
$\langle x-a \rangle$ { if $x > a$ as single
whole $x-a$ as single
var? if $x < a$ as -ve then $x-a < 0$

For values of x between

- (i) $0 < x < a$, only first term is considered
- (ii) $a < x < b$; only first 2 terms are considered.
- (iii) $b < x < L$; All terms are considered.

Q.1 Find ① y_c
② θ_A
③ $y_{max} \rightarrow \theta = 0$

$E = 200 \times 10^6 \text{ kN/m}^2$
 $I = 20 \times 10^{-6} \text{ m}^4$



→ Solⁿ:

For section ①-①
(L → R)

$M_x = 20 \langle x-0 \rangle$

$- 20 \langle x-1 \rangle$

$- 20$
 $- 10 \frac{\langle x-2 \rangle^2}{2}$

or $M_x = \frac{EI d^2 y}{dx^2}$

$= 20 \langle x-0 \rangle - 20 \langle x-1 \rangle$
 $- 10 \frac{\langle x-2 \rangle^2}{2}$

Integrating

$EI \frac{dy}{dx} = 20 \frac{x^2}{2} + C_1$

$- 20 \frac{\langle x-1 \rangle^2}{2}$

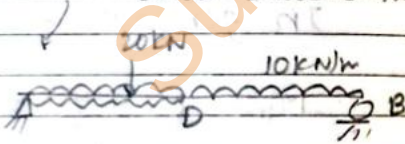
$- \frac{5}{3} \langle x-2 \rangle^3$ — (1)

x section can be taken
blw D & B { Analysis
from left

x If any lyth from st.
section blw A & C

But UDL should be taken
for whole beam
(s.t. UDL will be
in terms of x)

& inverse UDL for
balancing is added
below added section



Integrating

$EI y = \frac{10 x^3}{3} + C_1 x + C_2 - 10 \frac{\langle x-1 \rangle^3}{3}$

$- \frac{5 \langle x-2 \rangle^4}{12}$ — (1)

Boundary condition:-

$y_A = 0, y_B = 0$ if deflection

put, $y_A = 0, x = 0$

In (1)

$[C_2 = 0]$ negative $\langle x-1 \rangle$ is zero

Put, $y_B = 0, x = 4,$

In (1),

$C_1 = -175 = -29.167$
6

① Put $x = 1$, in (1)

$y_c = 6.458 \text{ mm}$

② Put, $x = 0$, in (1),

$\theta_A = 7.29 \times 10^{-3} \text{ rad.}$

Buckling :-

~~Classification of column~~

A long column when subjected to direct load deflects in lateral direction this phenomenon is called buckling.

Buckling takes place about the axis having least moment of inertia or minimum radius of gyration.

Slenderness ratio (λ):

It is the ratio of effective length to minimum of radius of gyration. It is the measure of buckling. $\lambda = \frac{L_{eff}}{r_{min}}$

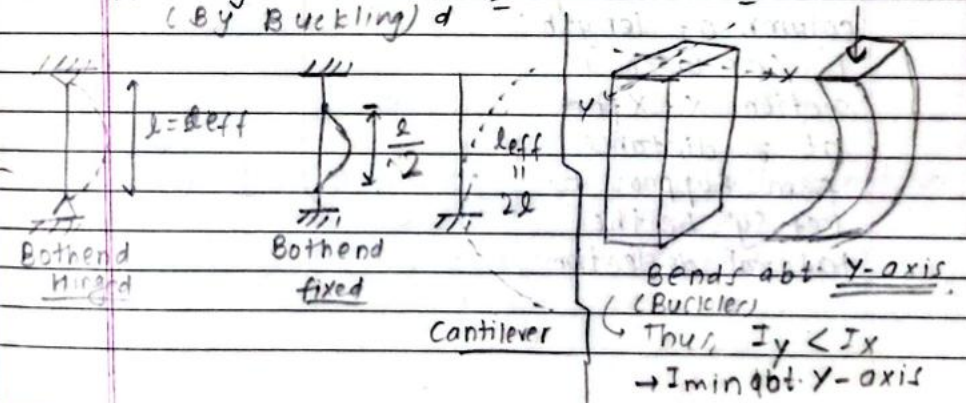
□ Classification of column

Depending upon slenderness ratio or length to diameter ratio: $d \rightarrow \text{dia. or dimen.}$

(1) short column $\frac{L}{d} < 8$ or $\lambda < 32$
(Failure by crushing)

(2) Intermediate column $8 < \frac{L}{d} < 30$ or $32 < \lambda < 120$
(By both buckling & crushing)

(3) Long column $\frac{L}{d} \geq 30$ or $\lambda \geq 120$
(By Buckling)

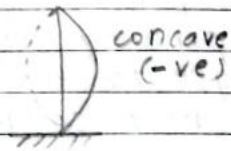
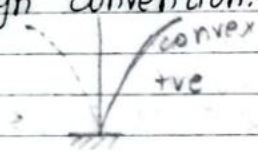


□ Analysis of long column: Euler's Theorem

• Assumptions:-

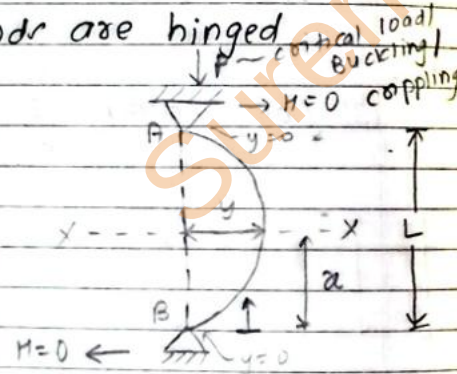
- (1) Column is initially straight and of uniform cross section.
- (2) Material of column is homogeneous and obeys Hooke's law.
- (3) Compressive load on column is axial. (passes thr CG & Eccentricity)
- (4) Weight of column is neglected.
- (5) Direct stress due to compressive force is very small as compared to bending stress due to buckling of column.
- (6) Column fails only by buckling. (long column)

□ Sign convention:-



□ Case I: Both ends are hinged

consider a long column of length 'L'. Consider a section x-x from support B. Let 'y' be the lateral deflection



eccentric column: load doesn't pass through CG, so, moment is developed (eccentricity)

Bending moment (BM) at section x-x = $-Py$
(abt. original position)

From Euler Bernoulli's equation,

$$M = EI \frac{d^2y}{dx^2} = -Py$$

$$\text{or, } EI \frac{d^2y}{dx^2} + Py = 0$$

$$\text{or, } \frac{d^2y}{dx^2} + \frac{Py}{EI} = 0 \quad [\lambda \rightarrow \text{not slender ratio}]$$

$$\text{or, } \frac{d^2y}{dx^2} + \lambda^2 y = 0 \quad \text{--- (I)}$$

$$[\text{Let, } \lambda^2 = \frac{P}{EI}]$$

This is homogeneous differential equation of 2nd order & its solution is

$$y = A \cos \lambda x + B \sin \lambda x \quad \text{--- (2)}$$

here A & B are constants of integration.

Boundary conditions:

$$\text{When } x=0, y=0 \Rightarrow A=0$$

$$\therefore y = B \sin \lambda x$$

$$\text{When } x=L, y=0$$

$$\Rightarrow B \sin \lambda L = 0$$

$$\text{Since, } A=0, B \neq 0$$

$$\text{So, } \sin \lambda L = 0$$

$$\therefore \lambda L = 0, \pi, 2\pi, \dots, n\pi$$

Not possible

$\lambda L = 0$ means $P=0$ So no deflection takes place. So, solutions considered are:

$$\lambda L = \pi, 2\pi, \dots, n\pi$$

$$\text{or, } \sqrt{\frac{P}{EI}} L = n\pi$$

$$\text{Or, } P = \frac{\pi^2 EI}{L^2}$$

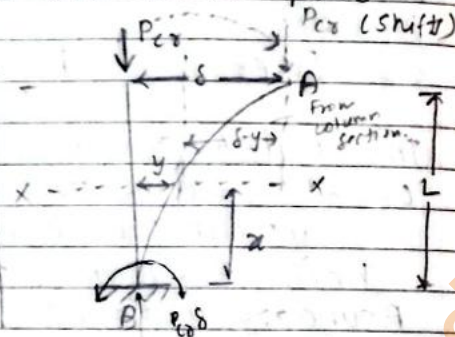
Smallest critical load is obtained when $n=1$.

$$\therefore P_{cr} = \frac{\pi^2 EI}{(L_{eff})^2}$$

$$\therefore L_{eff} = L \rightarrow \text{For Both end hinged.}$$

② One end fixed and other free

Consider a cantilever column of length L . Suppose column buckles due to critical load P_{cr} & δ be deflection at top



B.M. at section $x-x$ is

$$M_x = P_{cr}(\delta - y) \quad (\text{convex side})$$

Using Bernoulli's eqⁿ

$$EI \frac{d^2 y}{dx^2} = P_{cr}(\delta - y)$$

$$\text{Or, } EI \frac{d^2 y}{dx^2} + P_{cr} y = P_{cr} \delta$$

$$\text{Or, } \frac{d^2 y}{dx^2} + \frac{P_{cr}}{EI} y = \frac{P_{cr}}{EI} \delta$$

$$\text{Let, } \frac{P_{cr}}{EI} = \lambda^2$$

$$\frac{d^2 y}{dx^2} + \lambda^2 y = \lambda^2 \delta$$

Non-homogeneous differential equation of 2nd order and solution is

$$y = A \cos \lambda x + B \sin \lambda x + \delta \quad \text{--- (2)}$$

Boundary conditions

1) When $x=0, y=0$

$$\Rightarrow A + \delta = 0$$

$$\Rightarrow [A = -\delta]$$

② When $x=0, \frac{dy}{dx} = \theta_0 = 0$

$$\frac{dy}{dx} = -A \lambda \sin \lambda x + B \lambda \cos \lambda x$$

$$\frac{dy}{dx} \bigg|_0 = 0 \Rightarrow B \lambda = 0$$

$$\sin \lambda x, \lambda \neq 0, \theta = 0$$

$$\therefore y = -\delta \cos \lambda x + \delta$$

$$y = \delta(1 - \cos \lambda x) \quad \text{--- (3)}$$

When $x=L, y=\delta$

$$\Rightarrow \delta = \delta(1 - \cos \lambda L)$$

$$\Rightarrow \delta \cos \lambda L = 0$$

Since, $\delta \neq 0$ ($\because$ It means no deflection)

$$\therefore \cos \lambda L = 0$$

$$\lambda L = \frac{\pi}{2}, \frac{3\pi}{2}, \dots, \frac{n\pi}{2}$$

For, $n=1, \lambda L = \pi/2$

$$\sqrt{\frac{P_{cr} L}{EI}} = \pi/2$$

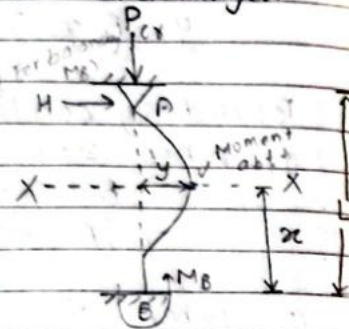
$$\therefore \left[P_{cr} = \frac{\pi^2 EI}{(2L)^2} = \frac{\pi^2 EI}{(L_{eff})^2} \right] u$$

Thus, $[L_{eff} = 2L] u$.

(3) One end fixed and other end hinged

→ Resisting moment M_B will be developed at fixed end B.

→ For the existence of M_B , a horizontal reaction H should be developed at hinge support



$$M_A = -P_{cr} \cdot y + H(L-x)$$

$$\text{or } EI \frac{d^2 y}{dx^2} = -P_{cr} y + H(L-x)$$

$$\text{or } \frac{d^2 y}{dx^2} + \frac{P_{cr}}{EI} y = \frac{H}{EI} (L-x)$$

$$\text{Let } \frac{P_{cr}}{EI} = \lambda^2 u$$

$$\therefore \frac{d^2 y}{dx^2} + \lambda^2 y = \frac{\lambda^2 H}{P_{cr}} (L-x) \quad \text{--- (1)}$$

Non-homogeneous differential eqⁿ of 2nd order and its solution is

$$y = A \cos \lambda x + B \sin \lambda x + \frac{H(L-x)}{P_{cr}} \quad \text{--- (2)}$$

$$\frac{dy}{dx} = -A \lambda \sin \lambda x + B \lambda \cos \lambda x - \frac{H}{P_{cr}} \quad \text{--- (3)}$$

Boundary conditions:

$$\text{(1) When } x=0, y=0 \Rightarrow A = \frac{-HL}{P_{cr}}$$

$$\text{(2) When } x=0, \frac{dy}{dx} = 0 \Rightarrow B = \frac{H}{\lambda P_{cr}}$$

$$\therefore y = \frac{-HL}{P_{cr}} \cos \lambda x + \frac{H}{\lambda P_{cr}} \sin \lambda x + \frac{H}{P_{cr}} (L-x)$$

$$\text{(3) When } x=L, y=0$$

$$\Rightarrow \frac{H}{P_{cr}} \left(\frac{\sin \lambda L}{\lambda} - L \cos \lambda L \right) = 0$$

$$\frac{H}{P_{cr}} \neq 0$$

$$\therefore \frac{\sin \lambda L}{\lambda} - L \cos \lambda L = 0$$

$$\text{or } \tan \lambda L = \lambda L \quad \text{--- (4)}$$

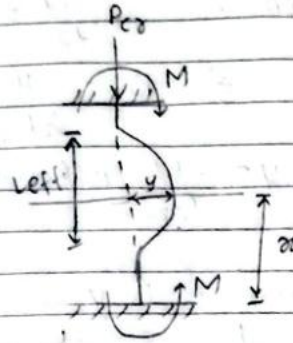
Solution of this equation is approximately $\lambda L = \sqrt{2} \pi$ rad

$$\text{or } \sqrt{\frac{P_{cr}}{EI}} L = \sqrt{2} \pi$$

$$\text{or } P_{cr} = \frac{\pi^2 EI}{\left(\frac{L}{\sqrt{2}}\right)^2} = \frac{\pi^2 EI}{(L_{eff})^2}$$

$$\text{Thus, } [L_{eff} = \frac{L}{\sqrt{2}}]$$

④ Both ends fixed



$$M_a = M - P_{cx} y$$

$$\text{or, } EI \frac{d^2 y}{dx^2} = M - P_{cx} y$$

$$\text{or, } \frac{d^2 y}{dx^2} + \frac{P_{cx}}{EI} y = \frac{M}{EI}$$

$$\text{Let, } \frac{P_{cx}}{EI} = \lambda^2$$

$$\text{or, } \frac{d^2 y}{dx^2} + \lambda^2 y = \frac{\lambda^2 M}{P_{cx}} \quad \text{--- (1)}$$

Non-homogenous differential equation of 2nd order and its solution is

$$y = A \cos \lambda x + B \sin \lambda x + \frac{M}{P_{cx}} \quad \text{--- (2)}$$

$$\frac{dy}{dx} = -A \lambda \sin \lambda x + B \lambda \cos \lambda x \quad \text{--- (3)}$$

Boundary conditions:-

$$\text{① When } x=0, y=0 \Rightarrow A = -\frac{M}{P_{cx}}$$

$$\text{② When } x=0, \frac{dy}{dx} = 0 \Rightarrow B = 0$$

$$\text{③ When } x=L, y=0 \Rightarrow -\frac{M}{P_{cx}} \cos \lambda L + \frac{M}{P_{cx}} = 0$$

$$\text{Compressive member: } \frac{M \lambda}{P_{cx}} \sin \lambda L + B \lambda \cos \lambda L = 0$$

$$\text{or, } \frac{M}{P_{cx}} (1 - \cos \lambda L) = 0$$

$$\therefore \frac{M}{P_{cx}} \neq 0$$

$$1 - \cos \lambda L = 0$$

$$\text{or, } \cos \lambda L = 1$$

$$\therefore \lambda L = 0, 2\pi, 4\pi, \dots, 2n\pi$$

considering least possible value,

$$\lambda L = 2\pi \quad \left\{ \text{Buckling starts for } 2\pi \right\}$$

$$\text{or, } \sqrt{\frac{P_{cx}}{EI}} L = 2\pi$$

$$\text{or, } P_{cx} = \frac{\pi^2 EI}{\left(\frac{L}{2}\right)^2} = \frac{\pi^2 EI}{(L_{eff})^2}$$

$$\text{Thus, } [L_{eff} = \frac{L}{2}]$$

□ Limitations of Euler's Formula

1. Applicable to an ideal column (or strut) only and in practice, the load applied may not be exactly co-axial.
2. It doesn't take account of direct stresses which means it may give a buckling load for struts far in excess of load which it can withstand under direct compression.
3. Euler's formula is applicable only for $\lambda > 80$.

$$P_{cx} = \frac{\pi^2 EI}{L_{eff}^2} \quad (\text{Mild steel})$$

$$\text{critical stress} = \frac{P_{cx}}{A} = \frac{\pi^2 EI}{A \cdot L_{eff}^2}$$

$$= \frac{\pi^2 E A r^2}{A \cdot L_{eff}^2} = \frac{\pi^2 E}{(L_{eff}/r)^2}$$

[MOI, $I = A r^2$
 $r \rightarrow$ rad. of gyration]

$$= \frac{\pi^2 E}{\lambda^2} \quad \lambda \rightarrow \text{Slenderness ratio}$$

$$[\sigma_c \propto \frac{1}{\lambda^2}]$$

For Euler's theorem, critical stress cannot be more than crushing stress because assumption is that column fails by buckling only.

$$\sigma_{cr} = \sigma_c = \frac{\pi^2 E}{\lambda^2}$$

For steel,

$$E = 2 \times 10^5 \text{ MPa}$$

$$\sigma_c = 315 \text{ MPa (yield point)}$$

$$\lambda = \sqrt{\frac{\pi^2 E}{\sigma_c}}$$

$$= 99.16 \approx 80$$

For $\lambda < 80$, $\sigma_c > 315 \text{ N/mm}^2$ (yield)

the material becomes plastic and no longer obeys Hooke's law.

Note:

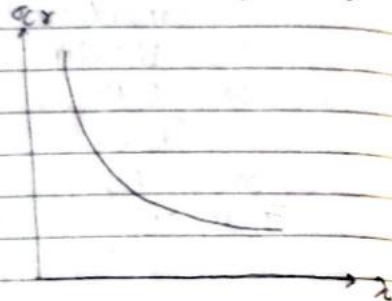
As $\lambda \rightarrow 0$, critical stress tends to infinity.

But this cannot happen. Before critical stress tends to infinity, material reaches crushing stress & gets crushed.

So, limiting value of critical stress is crushing stress.

$$(\sigma_{cr} = \sigma_{crushing})$$

$\lambda > 80 \rightarrow$ long column.



Rankine Formula

- For both short and long column (For intermediate column as well)
- Empirical formula (experimental)

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_E} \quad \text{--- (1)}$$

where, P is Rankine's critical load

P_c is crushing load

P_E is Euler's Buckling load

$$\left\{ \begin{aligned} P_E &= \frac{\pi^2 EI}{L_{eff}^2} ; P_c = \sigma_c \times A \\ P_E &= \frac{\pi^2 E}{\lambda^2} \end{aligned} \right.$$

For short column, $\frac{1}{P_E}$ becomes negligible

$$\text{Thus, } \frac{1}{P} = \frac{1}{\sigma_c \cdot A} + \frac{1}{\frac{\pi^2 EI}{(L_{eff})^2}}$$

$$\text{or } \frac{1}{P} = \frac{1}{\sigma_c \cdot A} + \frac{1}{\pi^2 EA \left(\frac{\sigma}{E}\right)^2}$$

$$\text{or, } \frac{A}{P} = \frac{1}{\sigma_c} + \frac{1}{\pi^2 E \left(\frac{\sigma}{E}\right)^2}$$

$$\text{or, } \frac{P}{A} = \frac{1}{\frac{1}{\sigma_c} + \frac{1}{\pi^2 E \left(\frac{\sigma}{E}\right)^2}}$$

$$\text{or, } \frac{P}{A} = \frac{\sigma_c}{1 + \frac{\sigma_c}{\pi^2 E} \left(\frac{L_e}{\sigma}\right)^2}$$

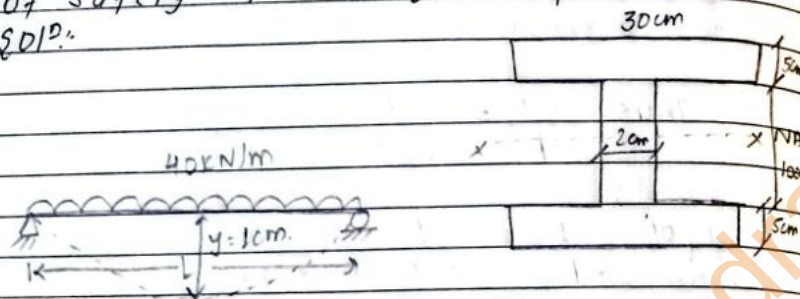
$$\text{or, } \left[P = \frac{\sigma_c A}{1 + \alpha \lambda^2} \right] \quad \begin{aligned} &\text{Rankine formula} \\ &\text{(For intermediate column)} \\ &[P = P_{\text{Rankine}}] \end{aligned}$$

$$\text{here, } \alpha = \frac{\sigma_c}{\pi^2 E} \rightarrow \text{Rankine constant}$$

Material	σ_c (MPa)	α
Mild steel	320	$1/7500$
Cast Iron	550	$1/1600$
Strong Timber	50	$1/750$

- Q. A builtup beam is simply supported. Calculate its length given that it is subjected to UDL of 40 kN/m and it deflects by 1 cm. Also, find out the safe load if this beam is used as a column with both ends fixed. Assume factor of safety 4 and $E = 210 \text{ GPa}$.

→ Solⁿ:

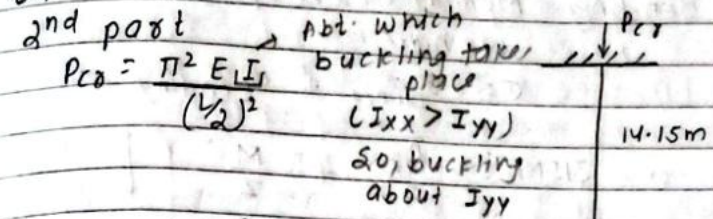


$$\begin{aligned}
 I_{xx} &= I_{\text{rect}(30 \times 110)} \\
 &\quad - I_{\text{rect}(28 \times 100)} \\
 &= \frac{30 \times 110^3}{12} - \frac{28 \times 100^3}{12} \\
 &= 9.942 \times 10^{-3} \text{ m}^4
 \end{aligned}$$

From double integration method,

$$\begin{aligned}
 y &= \frac{5}{384} \frac{WL^4}{EI} \\
 \text{or, } 1 \text{ cm} &= \frac{5}{384} \times \frac{40 \times L^4}{E I_{xx}} \quad \text{M.T. abt. NA} \\
 [L &= 14.15 \text{ m}]
 \end{aligned}$$

Q. 2.:



$$I = I_{yy} = \frac{2 \times 30^3 \times 5}{12} + \frac{100 \times 2^3}{12}$$

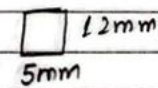
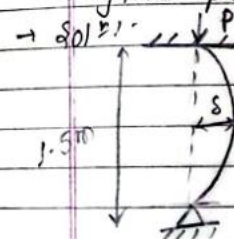
$$= 22566.67 \text{ cm}^4$$

$$P_{cr} = \frac{\pi^2 \times 2.1 \times 10^{11} \times 22566.67 \times 10^{-8}}{\left(\frac{14.15}{2}\right)^2}$$

$$= 9344 \text{ kN}$$

$$\text{Safe load, } = \frac{9344}{4 \text{ (F.S.)}} = 2336 \text{ kN}$$

- Q. A straight bar 1.5 m long and 12 mm x 5 mm in x-section is mounted on a test machine and is loaded until it buckles. Assume both ends are hinged. Estimate maximum central deflection before it attains yield point at 315 MPa (take $E = 2.1 \times 10^5 \text{ MPa}$).



$$\begin{aligned}
 \sigma_y &= 315 \text{ MPa} \\
 E &= 2.1 \times 10^5 \text{ MPa}
 \end{aligned}$$

$$\begin{aligned}
 P_{cr} &= \frac{\pi^2 EI}{(L_{eff})^2} \\
 &= \frac{\pi^2 \times 2.1 \times 10^{11} \times \frac{12 \times 5^3}{12} \times 10^{-12}}{(1.5)^2} \\
 &= 115.14 \text{ N}
 \end{aligned}$$

$$\sigma = \sigma_{\text{Direct}} + \sigma_{\text{Bending}}$$

(σ_{Max} so +)

$$\text{Bending Moment developed} = 115.14 \times 8$$

We have,

$$\sigma = \sigma_{\text{Direct}} + \sigma_{\text{Bending}}$$

$$\text{or } 315 \text{ MPa} = \frac{P_{\text{cr}}}{A} + \frac{M}{Z}$$

[Minimize y & buckling
if it happens]

$$\text{or } 315 \times 10^6 = \frac{115.14}{5 \times 12 \times 10^{-6}} + \frac{115.14 \times 8 \times 5 \times 10^{-3}}{\frac{12 \times 12^3 \times 10^{-12}}{12}}$$

$$\text{or } [8 = 326.3 \text{ mm}]$$

$$[8 = 135.96 \text{ mm}]$$

For rectangular section

$$I = \frac{bd^3}{12}$$

$$Z = \frac{bd^2}{6}$$

Date _____
Page _____

Beam Deflection

$$\frac{Wl^3}{48EI}$$

$$\frac{5}{384} \frac{Wl^3}{EI}$$

Stress Invariants

Characteristic equation is given by
 $\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$
 roots of this equation is unique & does not depend upon selection of co-ordinate axes

→ Stress matrix =

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

then,

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \text{minor of } \sigma_{xx} + \text{minor of } \sigma_{yy} + \text{minor of } \sigma_{zz}$$

$$I_3 = \text{Determinant of stress matrix}$$

Solve characteristic eqⁿ using calculator,
 $\sigma_1 > \sigma_2 > \sigma_3$

Q1

$$\left. \begin{array}{l} \sigma_x = 100 \text{ MPa} \\ \sigma_y = 60 \text{ MPa} \\ \tau_{xy} = -48 \text{ MPa} \end{array} \right\} \begin{array}{l} \sigma_1 = 132 \\ \sigma_2 = 28 \end{array}$$

Stress Matrix =

$$\begin{bmatrix} 100 & -48 & 0 \\ -48 & 60 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 160 \text{ MPa}$$

$$I_2 = \begin{vmatrix} 60 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 100 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 100 & -48 \\ -48 & 60 \end{vmatrix} = 3696 \text{ MPa}$$

$$I_3 = \begin{vmatrix} 100 & -48 & 0 \\ -48 & 60 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\sigma^3 - 160\sigma^2 + 3696\sigma - 0 = 0$$

$$\Rightarrow \sigma_1 = 132 \text{ MPa}, \sigma_2 = 28 \text{ MPa}, \sigma_3 = 0 \text{ MPa}$$

Q2 The principle stress at a point on two perpendicular planes is 80MPa & 40MPa. What will be the intensity of stress which acting along will produce the same maximum strain. If poisson's ratio, $\nu = \frac{1}{4}$.

→ Soln:

$\sigma_1 = 80 \text{ MPa}; \sigma_2 = 40 \text{ MPa}$
 Max. Stress (No Shear)
 Max. Strain along σ_1

$$\epsilon = \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E}$$

$$= \frac{80}{E} - \frac{1}{4} \frac{40}{E}$$

$$\left[\epsilon = \frac{70}{E} \right] \text{ Since } \epsilon = \frac{\sigma}{E} = \frac{70}{E}$$

$$[\sigma = 70 \text{ MPa}]$$

Q

Soln:

Extending UDL

for section upto its end s.t.

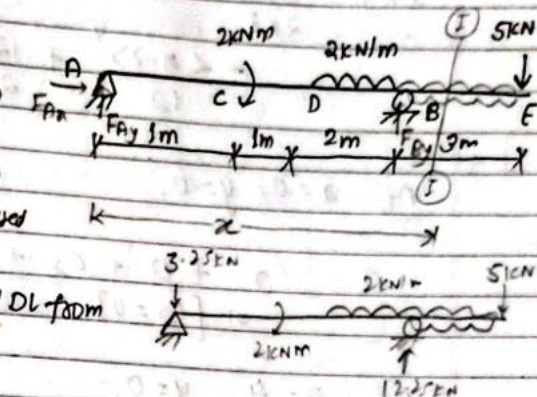
force/bxn due to

UDL can be expressed

in terms of α

& adding eq. UDL form

downward.



$$\sum M_A = 0 \quad (\uparrow +ve)$$

$$\text{or } 2 + 2 \times 2 \times 3 - F_{By} \times 4 + 5 \times 7 = 0$$

$$\Rightarrow [F_{By} = 12.25 \text{ kN}] (\uparrow)$$

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{or } F_{Ay} + 12.25 - 2 \times 2 - 5 = 0$$

$$\Rightarrow F_{Ay} = -3.25 \text{ kN}$$

$$[F_{Ay} = 3.25 \text{ kN} (\downarrow)]$$

NOW,

$$M_x = EI \frac{d^2y}{dx^2} = -3.25 \langle x-0 \rangle + 2 \langle x-1 \rangle^0$$

$$- 2 \langle x-2 \rangle \langle x-2 \rangle + 12.25 \langle x-4 \rangle$$

$$+ \frac{2 \langle x-4 \rangle^2}{2} \quad \text{--- (1)}$$

Integrating.

$$EI \frac{dy}{dx} = -3.25 \langle x-2 \rangle + 12.25 \langle x-4 \rangle + \frac{2 \langle x-4 \rangle^3}{3}$$

$$EI \frac{dy}{dx} = -3.25 \langle x-0 \rangle^2 + \frac{2 \langle x-1 \rangle^3}{3} + \frac{12.25 \langle x-4 \rangle^2}{2} + \frac{2 \langle x-4 \rangle^3}{3} \quad \text{--- (2)}$$

$$EIy = \frac{-3.25}{2 \times 3} \langle x-0 \rangle^3 + C_1 x + C_2 + \frac{\langle x-4 \rangle^2}{2 \times 3} - \frac{\langle x-2 \rangle^4}{12} + \frac{12.25}{2 \times 3} \langle x-4 \rangle^3 + \frac{\langle x-4 \rangle^4}{12}$$

At, $x=0, y=0,$

$$0 = 0 + 0 + C_2 + 0 + 0 + 0 + 0$$

$$\Rightarrow [C_2 = 0]$$

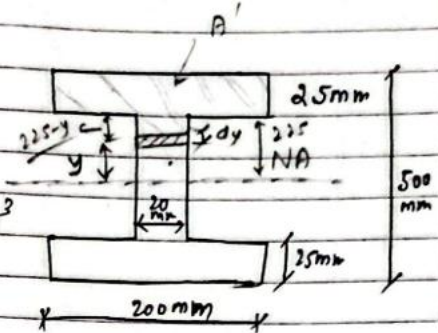
At, $x=4, y=0,$

9. An I-section beam is subjected to a shear force 'F', find what % of shear is resisted by web.

→ Solⁿ:
MOI of section about neutral axis (N.A.), I

$$= \frac{200 \times 500^3}{12} - \frac{180 \times 450^3}{12}$$

$$= 7.1646 \times 10^8 \text{ mm}^4$$



Consider an element in web at a distance y from N.A. [Dividing A' into two parts → web & flange]
Depth of web = $225 - y$

$$CG \text{ from N.A.} = y + \frac{225-y}{2} = \frac{y+225}{2}$$

By mo p. 20

dF = τdA

$= I / (2013)$

200

20

500

element

$$A' \bar{y} = (A' \bar{y}) \text{ of flange} + (A' \bar{y}) \text{ of web}$$

$$= [200 \times 25 \times (250 - 12.5)] + [20 \times (225 - y) \times \frac{225 + y}{2}]$$

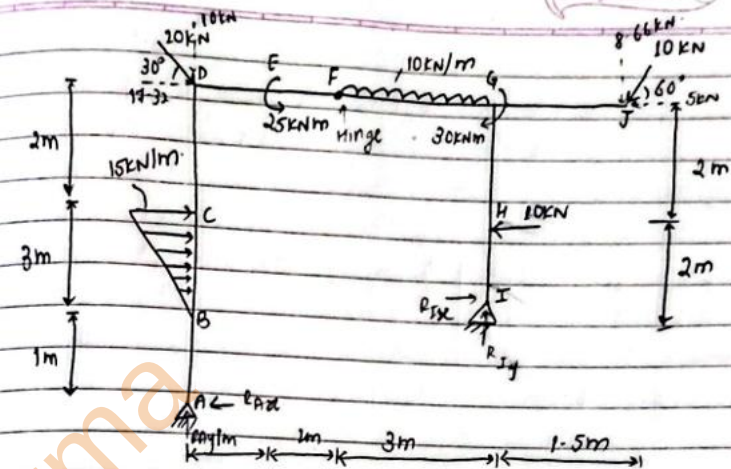
$$= 1187500 + 10(225^2 - y^2)$$

$$\text{Shear stress at } y, \tau = \frac{VA' \bar{y}}{I_b}$$

$$\tau = \frac{F [1187500 + 10(225^2 - y^2)]}{7.1646 \times 10^8 \times 20}$$

$$\text{For } \theta, F_{web} = \int_{-225}^{225} \tau \times dA = \int_{-225}^{225} \frac{F [1187500 + 10(225^2 - y^2)]}{7.1646 \times 10^8 \times 20} \times 20 dy$$

$$[F_{web} = 95.78\% F]$$



Calculation of unknown reactions:-

$$\sum M_{\text{left of hinge}} = 0 \quad (\uparrow +ve)$$

$$\text{or, } 25 + 10 \times 3 + 15 \times 3 \times \frac{1}{2} \times \left(2 + \frac{1}{3} \times 3\right) - R_{Ay} \times 3 - R_{Ax} \times 6 = 0$$

$$\text{or, } 6R_{Ax} + 3R_{Ay} = 197.5 \quad \text{--- (i)}$$

$$\sum M_A = 0 \quad (\uparrow +ve)$$

$$\text{or, } -R_{Ay} \times 6 - R_{Ax} \times 2 - \frac{1}{2} \times 15 \times 3 \times \left(\frac{2}{3} \times 3 + 1\right)$$

$$\sum F_x = 0 \quad (\rightarrow +ve)$$

$$\text{or, } -R_{Ax} + \frac{1}{2} \times 15 \times 3 + 17.32 - 5 - 10 + R_{Ix} = 0$$

$$R_{Ix} - R_{Ax} = 24.82 \quad \text{--- (ii)}$$

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{or, } R_{Ay} - 10 - 10 \times 3 - 8.66 + R_{Iy} = 0$$

$$\Rightarrow R_{Ay} + R_{Iy} = 48.66 \quad \text{--- (iii)}$$

$$\sum M_{\text{right of hinge}} = 0 \quad (\uparrow +ve)$$

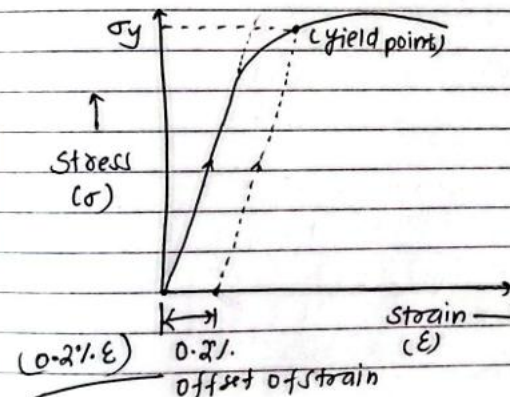
$$\text{or, } -10 \times 3 \times 1.5 - 30 - 10 \times 2 - 8.66 \times 4.5$$

$$\Rightarrow 3R_{Iy} + 4R_{Ix} = 133.95 + R_{Iy} \times 3 + R_{Ix} \times 4 = 0$$

Beyond yield point $\Rightarrow$ plastic deformation starts.

Offset Method:

- $\hookrightarrow$ used for such material which do not have pronounced yield strength
- $\hookrightarrow$ For aluminium and other ductile material horizontal portion of stress-strain curve doesn't characterize yield point.
- $\hookrightarrow$ For brittle material, yield and necking are not significant & material eventually fractures (rupture occurs without noticeable prior change in the rate of elongation. Strain at time of rupture is much smaller for brittle than ductile materials)
- $\hookrightarrow$ For such material yield strength determined by offset method.
- $\hookrightarrow$ The yield strength at 0.2% offset of strain is obtained by drawing, through the point of horizontal axis of abscissa $\epsilon = 0.2\%$ ($\epsilon = 0.002$), a line parallel to initial straight line portion of stress-strain diagram.
- $\hookrightarrow$ Stress (σ_y) obtained is defined as yield strength at 0.2% offset of strain.



Parallel axis Theorem,

$$I_{AA} = I_G + A b^2$$

Polar moment of inertia

$$J_O = \int r^2 dA = I_y + I_x$$



Radius of gyration of an area

If area is concentrated in a thin strip parallel to x-axis,

$$I_x = k_x^2 A$$

$$k_x = \sqrt{\frac{I_x}{A}} \text{ (Rad. of gyration w.r.t. x-axis)}$$

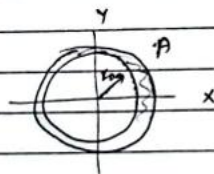


Similarly,

$$I_y = k_y^2 A$$

$$J_O = k_O^2 A$$

$$k_O^2 = k_x^2 + k_y^2$$



Product of inertia

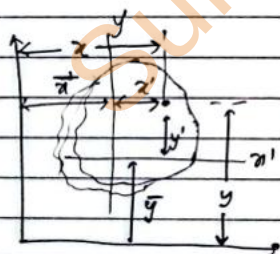
$$I_{xy} = \int xy dA$$

When x-axis, y-axis or both are axis of symmetry,

then POI = 0

Parallel axis theorem,

$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$$



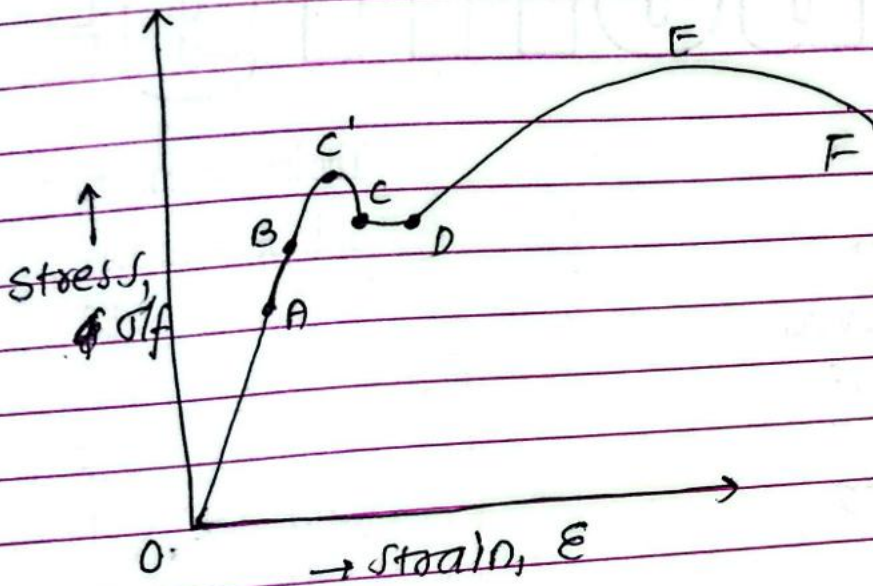
principle axes & principle moments of inertia

$$I_{avg} = \frac{I_x + I_y}{2}$$

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$(I_{x'} - I_{avg})^2 + I_{x'y'}^2 + R^2$$

IMPORTANT NOTES



OAB → represents st. line

(obeys Hooke's law)

A → limit of proportionality

B → elastic limit

C'/C → represents the upper/lower yield point

CD → plastic yielding

DE → strain hardening

E → Ultimate stress

EF → necking

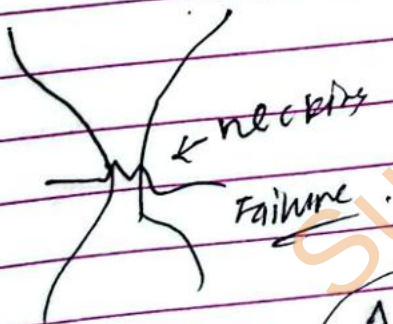
F → breaking stress

stress-strain relation for
Mild steel

Young's Modulus → 'E'

Yield point → σ_{yl} / f_y → yield stress

250 MPa for MS



After this, MS goes to plastic deformation.

$$\Delta L = \frac{PL}{AE}$$